CSC236 tutorial exercises

(Best before 11 am, Monday October 15th)

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Here are your tutorial sections:

Surname	Section	Room	TA
A–F	Day 1 (11:00 am)	LM162	Yuval
G–Li	Day 2 (11:00 am)	BA2139	Lila
Lo–Si	Day 3 (11:00 am)	BA2145	Oles
So–Z	Day 4 (11:00 am)	BA2155	Lalla
A–H	Evening 1 (8:00 pm)	BA1190	Colin
I–M	Evening 2 (8:00 pm)	BA2135	Norman
N–Z	Evening 3 (8:00 pm)	BA2139	Feyyaz

These exercises are intended to give you practice with unwinding and proving recurrences.

1. Consider the recurrence:
$$T(n) = \begin{cases} 1 & \text{if } n = 1 \\ 1 + T(\lceil n/2 \rceil) & \text{if } n > 1 \end{cases}$$
Use complete induction to prove that for every positive natural number n, $T(n) \geq c\lg(n)$, for some positive real constant c.

2. Unwind the recurrence from the previous question in the case where $n = 2^k$ for some positive integer k (see annotated slides from October 11th or 12th). Use mathematical induction on k to prove that $T(2^k) = k + 1$.

3. Consider another recurrence:
$$G(n) = \begin{cases} 1 & \text{if } n < 2 \\ 1 + G(n-1) + G(n-2) & \text{if } n \geq 2 \end{cases}$$
Unwind the recurrence **carefully**, following the pattern below, for some n that is comfortably greater than 1:
$$\begin{aligned} G(n) &= 1 + G(n-1) + G(n-2) \\ &= 1 + (1 + G(n-2) + G(n-3)) + G(n-2) = 2 + 2G(n-2) + G(n-3) \\ &= 2 + 2(1 + G(n-3) + G(n-4)) + G(n-3) = 4 + 3G(n-3) + 2G(n-4) \\ &= 4 + 3(1 + G(n-4) + G(n-5)) + 2G(n-4) = 7 + 5G(n-4) + 3G(n-5) \\ &\vdots \end{aligned}$$
Can you see a pattern that leads to a guess at a closed form for $G(n)$?

Solutions

1. $P(n) : T(n) \geq c\log(n)$ (for some positive constant c that we determine below).

   Induction step: Assume n is a generic positive natural number, and that $P(i)$ for every natural number i, $1 \leq i < n$.

   Case $n = 1$ (base case): If $n = 1$, then $T(n) = 1 \geq c0$ for any real number c, so $P(1)$ is verified without recourse to induction.

   Case $n > 1$: Then, by definition $T(n) = 1 + T(\lceil n/2 \rceil)$, so

$$\begin{aligned} T(n) &= 1 + T(\lceil n/2 \rceil) \\ &\geq 1 + c\lg(\lceil n/2 \rceil) \qquad \text{by IH, since } 1 \leq \lceil n/2 \rceil < n, n > 1. \\ &\geq 1 + c\lg(n/2) \qquad \text{since lg is increasing} \\ &= 1 + c(\lg(n) - \lg(2)) = c\lg(n) + 1 - c \\ &\geq \geq c\lg(n) \qquad \text{provided } 1 \geq c. \end{aligned}$$

   So $T(n) \geq c\lg(n)$ holds in both possible cases.

   So, for every positive integer n, if $P(i)$ is true for $1 \leq i < n$, then so is $P(n)$.

   Conclude $P(n)$ for all positive natural numbers, by Complete Induction.

   The inequality $1 \leq \lceil n/2 \rceil < n$ for $n > 1$ is proved in the Course Notes.

2. $P(k) : T(2^k) = 1 + k$. Proof of that $\forall k \in \mathbb{N}, P(k)$ by Mathematical Induction.

   Base case ($k = 0$): Then $T(2^k) = T(1) = 1 = 1 + k$, so the claim is verified for $k = 0$.

   Induction step: Assume that k is a generic natural number and that $P(k)$ holds, that is $T(2^k) = 1 + k$. Then

$$\begin{aligned} T(2^{k+1}) &= 1 + T(\lceil 2^k \rceil) \qquad \text{definition, since } 2^{k+1} > 1 \\ &= 1 + T(2^k) \qquad \lceil 2^k \rceil = 2^k \\ &= 1 + 1 + k \qquad \text{by IH} \\ &= 1 + (k + 1) \end{aligned}$$

   That is, $P(k + 1)$.

   So, $\forall n \in \mathbb{N}, P(k) \Rightarrow P(k + 1)$

   Conclude, $\forall k, P(k)$, by Mathematical Induction.

3. Unwinding farther, and noticing some similarity to the Fibonacci numbers $F(i)$:

$$\begin{aligned}
G(n) &= 1 + G(n-1) + G(n-2) \\
&= 1 + (1 + G(n-2) + G(n-3)) + G(n-2) = 2 + 2G(n-2) + G(n-3) \\
&= 2 + 2(1 + G(n-3) + G(n-4)) + G(n-3) = 4 + 3G(n-3) + 2G(n-4) \\
&= 4 + 3(1 + G(n-4) + G(n-5)) + 2G(n-4) = 7 + 5G(n-4) + 3G(n-5) \\
&= 7 + 5(1 + G(n-6) + G(n-5)) + 3G(n-5) = 12 + 8G(n-5) + 5G(n-6) \\
&\vdots \\
&= [F(i+1) - 1] + F(i)G(n-i+1) + F(i-1)G(n-i) \\
&\vdots \\
&= [F(n+1) - 1] + F(n)G(1) + F(n-1)G(n-n) = 2F(n+1) - 1
\end{aligned}$$