CSC236 tutorial exercises, Week #2

(Best before 11 am, Monday October 1st)

Danny Heap

Here are your tutorial sections:

Surname	Section	Room	TA
A–F	Day 1 (11:00 am)	LM162	Lila
G–Li	Day 2 (11:00 am)	BA2139	Yuval
Lo–Si	Day 3 (11:00 am)	BA2145	Oles
So–Z	Day 4 (11:00 am)	BA2155	Lalla
A–H	Evening 1 (8:00 pm)	BA1190	Colin
I–M	Evening 2 (8:00 pm)	BA2135	Norman
N–Z	Evening 3 (8:00 pm)	BA2139	Feyyaz

These exercises are intended to give you practice with complete induction, proving inequalities, and dealing with cases where the base cases aren't obvious.

1. Recall the definition of a full binary tree from the annotated lecture slides or the course notes, example 1.13, page 42. Use Complete Induction to prove that every non-empty full binary tree has exactly one more leaf than interior nodes.

2. Use Complete Induction, and emulate the course notes, example 1.12, page 40 to show that postage of exactly n cents can be made using only 3-cent and 5-cent stamps, for every natural number n greater than k (you will have to discover the value of k).

3. Use Mathematical Induction to prove that for all natural numbers n, $n^4 \leq 4^n + 17$.

Sample solutions

1. $P(n)$: Every full binary tree with n nodes has exactly one more leaf than interior nodes. I will prove that $P(n)$ is true for every positive number n.

   Proof (by Complete Induction)

   Induction step: Assume that n is a positive natural number, and that for every natural number $0 < i < n$, $P(i)$ is true.

   Case 1: $n = 1$: A full binary tree with 1 node has one leaf (the root) and zero interior nodes, so $P(1)$ holds. (Notice that this is the base case, since it is verified independently of the induction hypothesis).

   Case 2: $n > 1$: Since a full binary tree with more than 1 nodes must have nodes other than the root, the root has 2 children (definition of full binary tree). The two children, in turn, are roots of non-empty full binary trees, since removing a parent doesn't change the properties that make them trees, binary, or full. Call these subtrees T_L and T_R, with n_L and n_R nodes, respectively. Since T_L and T_R are non-empty trees, n_L and n_R are greater than 0. Since T_L and T_R contain strict subsets of the nodes of the original tree (they lack the root), n_L and n_R are less than n. So $0 < n_L, n_R < n$, and so by the induction hypothesis T_L has i_L interior nodes and $i_L + 1$ leaves, and T_R has i_R interior nodes and $i_R + 1$ leaves. The interior nodes of T_R and T_L, plus the root, are the interior nodes of the original tree, so this tree has $i_L + i_R + 1$ interior nodes versus $i_L + 1 + i_R + 1$ leaves. In other words, exactly one more leaf than interior nodes, and $P(n)$ holds.

   Since n was assumed to be a generic positive natural number, $\forall n \in \mathbb{N} - \{0\}$, if $P(i)$ is true for every $0 < i < n$, then $P(n)$.

   I conclude that for all positive natural numbers n, $P(n)$.

2. $P(n)$: Postage of exactly n cents can be formed using only 3-cent and 5-cent stamps. I will prove that $P(n)$ is true for every natural number n greater than 7.

   Proof (by Complete Induction)

   Induction step: Assume that n is a generic natural number greater than 7 and that if $7 < i < n$, then $P(i)$ is true.

   Case 1, $n \in \{8, 9, 10\}$: Then postage of n cents can be formed with a 3-cent and a 5-cent stamp, or three 3-cent stamps, or with 2 5-cent stamps, so $P(n)$ follows.

   Case 2, $n > 10$: Then $7 < n - 3 < n$, and the induction hypothesis says that postage of exactly $n - 3$ cents can be formed using only 3-cent and 5-cent stamps. By adding a 3-cent stamp to this postage we have postage of n cents using only 3-cent and 5-cent stamps, that is $P(n)$.

   In both possible cases, $P(n)$ follows.

   Since n is assumed to be a generic natural number greater than 7, then $\forall n \in \mathbb{N} - \{0, 1, 2, 3, 4, 5, 6, 7\}$, if $P(i)$ is true for every $7 < i < n$, then $P(n)$ is also true.

   I conclude that for every natural number n greater than 7, $P(n)$, by Complete Induction.

3. $P(n) : 4^n + 17 \geq n^4$. I will prove that $P(n)$ is true for every natural number.

   Proof (by Mathematical Induction):

Base cases: $n < 4$: It is straightforward to verify $P(n)$ for $n \in \{0, 1, 2, 3\}$. Notice that the related inequality, $4n^4 - 51 \geq (n+1)^4$ (see below) is **false** for $n \in \{0, 1, 2\}$, so none of these four values can be reached using the logic of the induction step — we require all four base cases.

Induction step: Assume $n \in \mathbb{N} - \{0, 1, 2\}$ and that $P(n)$ is true.

Then

$$\begin{aligned}
4^{n+1} + 17 = 4 \times 4^n + 17 &\geq 4n^4 - 51 \quad \# \quad \text{by IH } 4 \times (4^n + 17) \geq 4n^4 \\
&= n^4 + n^4 + n^4 + n^4 - 51 \\
&\geq n^4 + 3n^3 + 3n^3 + 3n^3 - 51 \quad \# \quad n \geq 3 \Rightarrow n^4 \geq 3n^3 \\
&= n^4 + 4n^3 + 2n^3 + n^3 + 2n^3 - 51 \quad \# \quad \text{re-write} \\
&\geq n^4 + 4n^3 + 6n^2 + 4n + 2n^3 - 51 \\
&\qquad \# \quad n \geq 3 \Rightarrow 2n^3 \geq 6n^2 \wedge n^3 = n^2 n \geq 4n \\
&\geq n^4 + 4n^3 + 6n^2 + 4n + 1 \quad \# \quad n \geq 3 \Rightarrow 2n^3 - 51 \geq 1 \\
&= (n+1)^4
\end{aligned}$$

So $P(n+1)$ follows

Since I assumed n is a generic natural number greater than 2, $\forall n \in \mathbb{N} - \{0, 1, 2\}, \; P(n) \Rightarrow P(n+1)$.

I conclude $\forall n \in \mathbb{N}, P(n)$, by Mathematical Induction.