CSC236 fall 2012

recursion, induction, correctness

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Using Introduction to the Theory of Computation, Chapter 2

Outline

binary search

recursive binary search

```
def recBinSearch(x, A, b, e) :
  if b == e :
    if x <= A[b] :
      return b
    else :
      return e + 1
  else :
    m = (b + e) // 2 # midpoint
    if x <= A[m] :
      return recBinSearch(x, A, b, m)
    else :
      return recBinSearch(x, A, m+1, e)
```

no loops.

conditions, pre- and post-

- x and elements of A are comparable ✓
- e and b are valid indices, $b \leq e$ ✓ $e \geq 0$
- $A[b..e]$ is sorted non-decreasing

RecBinSearch(x, A, b, e) terminates and returns index p

- $b \leq p \leq e + 1$
- $b < p \Rightarrow A[p-1] < x$ ✓
- $p \leq e \Rightarrow x \leq A[p]$ ✓

(except for boundaries, returns p so that $A[p-1] < x \leq A[p]$)

precondition $\Rightarrow$ termination and postcondition

Proof: induction on $n = e - b + 1$

Base case, $n = 1$: Terminates because there are no loops or further calls, returns $x \leq A[b = p] \Leftrightarrow p = b = e$ is returned. $x > A[b = p - 1] \Leftrightarrow p = b + 1$ returned, so postcondition satisfied. Notice that the choice forces if-and-only-if.

Induction step: Assume $n > 1$ and that the postcondition is satisfied for inputs of size $1 \leq k < n$ that satisfy the precondition. Call RecBinSearch(A,x,b,e) when $n = e - b + 1 > 1$. Since $b < e$ in this case, the test on line 1 fails, and line 7 executes. Exercise: $b \leq m < e$ in this case. There are two cases, according to whether $x \leq A[m]$ or $x > A[m]$.

$m = \left\lfloor \frac{e+b}{2} \right\rfloor$

Case 1: $x \leq A[m]$

$\underbrace{0 < m-b+1}_{\text{by ex}} \underbrace{<}_{\text{by ex}} e-b+1$ So

- Show that $\boxed{\text{IH applies to RBS(x,A,b,m)}}$
- Translate the postcondition to RBS(x,A,b,m), IH tells us.
  - $b \leq p \leq m+1$
  - $b < p \Rightarrow A[p-1] < x$
  - $p \leq m \Rightarrow x \leq A[p]$

  by Induct. Hypoth.
- Show that RBS(x,A,b,e) satisfies postcondition
  - By IH: $b \leq p \leq m+1 \leq e < e+1$ ✓
  - $b < p \Rightarrow A[p-1] < x$ # by IH
  - $p \leq e \Rightarrow x \leq A[p]$ must show $x \leq A[p]$
    - if $p = m+1$, then $x \leq A[m]$ by $A[m] \leq A[m+1] = A[p]$ by A
    - if $p \leq m$, then $x \leq A[p]$ by IH

Case 2: $x > A[m]$

exercise.

- Show that IH applies to RBS(x,A,m+1,e)
- Translate postcondition to RBS(x,A,m+1,e), IH means.

• $b < m+1 \le p \le e+1$ ✓
• $p \le e \Rightarrow x \le A[p]$
• $m+1 < p \Rightarrow A[p-1] < x$

- Show that RBS(x,A,b,e)

• $b \le p \le e+1$ # by IH and $m \ge b$
• $p \le e \Rightarrow x \le A[p]$ # by IH.
• Show $b < p \Rightarrow A[p-1] < x$
  • $p > m+1$, then $A[p-1] < x$ by IH
  • $p \le m+1 \Rightarrow p = m+1$, then $x > A[m] = A[p-1]$ by

what could go wrong?

- $m = \lceil \frac{e+b}{2.0} \rceil$ prove or counterexample,
- $x < A[m]$ — prove or counter example
- ...
- Either prove correct, or find a counter-example