CSC236 fall 2012

more complexity: mergesort

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Using Introduction to the Theory of Computation, Chapter 3

Outline

divide and conquer (recombine)

using the Master Theorem

Notes

General case

revisit. . .

Class of algorithms: partition problem into b *roughly* equal subproblems, solve, and recombine:

$$T(n) = \begin{cases} k & \text{if } n \leq B \\ a_1 T(\lceil n/b \rceil) + a_2 T(\lfloor n/b \rfloor) + f(n) & \text{if } n > B \end{cases}$$

where $B, k > 0$, $a_1, a_2 \geq 0$, and $a_1 + a_2 > 0$. $f(n)$ is the cost of splitting and recombining.

Master Theorem

(for divide-and-conquer recurrences)

If f from the previous slide has $f \in \theta(n^d)$, then

$$T(n) = \begin{cases} \theta(n^d) & \text{if } a < b^d \\ \theta(n^d \log n) & \text{if } a = b^d \\ \theta(n^{\log_b a}) & \text{if } a > b^d \end{cases}$$

Proof sketch

1. Unwind the recurrence, and prove a result for $n = b^k$
2. Prove that T is non-decreasing
3. Extend to all n, similar to MergeSort

multiply lots of bits

what if they don't fit into a machine instruction?

$$\begin{array}{r} 1101 \\ \times 1011 \\ \hline \end{array}$$

divide and recombine

recursively. . .

11	01
$\times$10	11

$$xy = 2^n x_1 y_1 + 2^{n/2}(x_1 y_0 + y_1 x_0) + x_0 y_0$$

compare costs

n n-bit additions versus:

1. divide each factor (roughly) in half
2. multiply the halves (recursively, if they're too big)
3. combine the products with shifts and adds

Gauss's trick

$$xy = 2^n x_1 y_1 + x_0 y_0 + 2^{n/2} \left((x_1 + x_0)(y_1 + y_0) - x_1 y_1 - x_0 y_0 \right)$$

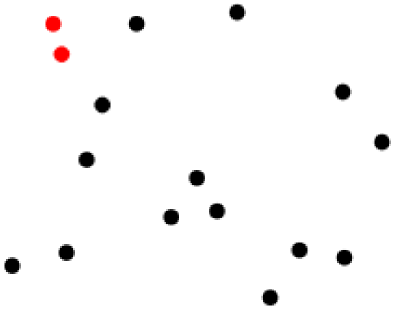
Gauss's payoff

lose one multiplication

1. divide each factor (roughly) in half
2. sum the halves
3. multiply the sum and the halves Gauss-wise
4. combine the products with shifts and adds

closest point pairs

see Wikipedia



divide-and-conquer v0.1

how many close points fit?

an $n \lg n$ algorithm

P is a set of points

1. Construct (sort) P_x and P_y
2. For each recursive call, construct L_x, L_y, R_x, R_y
3. Recursively find closest pairs (l_0, l_1) and (r_0, r_1), with minimum distance δ
4. V is the vertical line splitting L and R, D is the δ-neighbourhood of V, and D_y is D ordered by y-ordinate
5. Traverse D_y looking for mininum pairs 15 places apart
6. Choose the minimum pair from D_y versus (l_0, l_1) and (r_0, r_1).

Notes