Office Hour:
Friday 3:30-5
in BA3201

Tutorial sample
solution here!

CSC236 fall 2012
structural induction

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Using Introduction to the Theory of Computation,
Chapter 4, Section 1.1

Computer Science
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Outline

Equivalence of inductions

Structural induction

Notes

The cycle is proved in the text, here is one link. Suppose you believe MI, and you have shown for some property P:

$$\forall n \in \mathbb{N}, (\forall 0 \le i \le n, P(i)) \Rightarrow P(n) \qquad (1)$$

Now define a slightly different predicate:

$P'(n) : \forall 0 \le i \le n, P(i)$, in other words, $P(i)$ is true up to and including n. Using only MI prove $\forall n, P'(n)$:

Base case: Since we showed (1), and there are no natural numbers smaller than 0, we have $P'(0)$.

Induction step: Assume n is an arbitrary natural number and that $P'(n)$ is true. It follows from (1) that $P(n+1)$ is true, and hence $P'(n+1)$ is true.

Define sets inductively

... so as to use induction on them later

One way to define the natural numbers:

$\mathbb{N}$: The **smallest** set such that

1. $0 \in \mathbb{N}$
2. $n \in \mathbb{N} \Rightarrow n + 1 \in \mathbb{N}$.

By **smallest** I mean $\mathbb{N}$ has no proper subsets that satisfy these conditions. If I leave out **smallest**, what other sets satisfy the definition?

$\mathbb{Q}, \mathbb{R}, \mathbb{C},$

What can you do with it?

The definition on the previous page defined the simplest natural number (0) and the rule to produce new natural numbers from old (add 1). Proof using Mathematical Induction work by showing that 0 has some property, and then that the rule to produce natural numbers preserves the property, that is

1. Prove $P(0)$

2. Prove that $\forall n \in \mathbb{N}$, $P(n) \Rightarrow P(n+1)$.

Other structurally-defined sets

$\mathcal{E}$ includes $\quad x, \ y, \ z \quad (x+y) \quad (y+z)$

$(y+x), \ (y \times x) \quad ((x+y) \times (y+z))$

$(x \times (y+z))$

Define $\mathcal{E}$: The **smallest** set such that

- $x, y, z \in \mathcal{E}$
- $e_1, e_2 \in \mathcal{E} \Rightarrow (e_1 + e_2), (e_1 - e_2), (e_1 \times e_2),$
  and $(e_1 \div e_2) \in \mathcal{E}$.

Form some expressions in $\mathcal{E}$. Count the number of variables (symbols from $\{x, y, z\}$) and the number of operators (symbols from $\{+, \times, \div, -\}$). Make a conjecture.

$vr(e) = op(e) + 1$

Structural induction

$\left(\left(x + y \right) + z \right)$

$P(e):\ \text{vr}(e) = \text{op}(e) + 1$

To prove that a property is true for all $\underline{e \in \mathcal{E}}$, parallel the recursive set definition:

▶ **Base case:** Show that the property is true for the simplest members, $\{x, y, z\}$

▶ **Induction step:** Show "inheritance": if $P(e_1)$ and $P(e_2)$, then all possible combinations $(e_1 + e_2)$, $(e_1 - e_2)$, $(e_1 \times e_2)$, and $(e_1 \div e_2)$ have the property.

Conclude that the property is true of all elements of $\mathcal{E}$.

Structural induction

Conclude $\forall e \in \mathcal{E}, P(e)$, by SI

$P(e): \text{vr}(e) = \text{op}(e) + 1$

\# variables   \# operations

Prove $\forall e \in \mathcal{E}, P(e)$

__Proof__ (structural induction).

__Base case__ for basis elements $x, y, z \in \mathcal{E}$, each has $\text{vr}(e) = 1 = 0 + 1 = \text{op}(e) + 1$, so claim holds base case.

__Induction step__ IH Assume $P(e_1)$ and $P(e_2)$ hold for some arbitrary $e_1, e_2 \in \mathcal{E}$. Denote $\odot \in \{+, -, \times, \div\}$ and note that $\text{vr}((e_1 \odot e_2))$ $= \text{vr}(e_1) + \text{vr}(e_2)$. Also note $\text{op}((e_1 \odot e_2)) = \text{op}(e_1) + \text{op}(e_2) + 1$. That means

$$\text{op}((e_1 \odot e_2)) = \text{op}(e_1) + \text{op}(e_2) + 1$$
$$= \text{vr}(e_1) - 1 + \text{vr}(e_2) - 1 + 1 \quad (\text{by IH})$$
$$= \text{vr}(e_1) + \text{vr}(e_2) - 1 = \text{vr}((e_1 \odot e_2)) - 1,$$

We have shown $P((e_1 \odot e_2))$

Recursive definition

Fibonacci sequence

$gcd(m, n)$ → 0 | 1 | 1 | 2 |

This sequence comes up in applied rabbit breeding, the height of <u>AVL</u> trees, and the complexity of Euclid's algorithm for the GCD:

$$F(n) = \begin{cases} n & n < 2 \\ F(n - 2) + F(n - 1) & n \geq 2 \end{cases}$$

$n+1$, $n+2$

What is the sum of n Fibonacci numbers?

0 | 1 | 1 | 2 | 3 | 5 | 8 | 13 | 21 34

0 | 1 | 2 | 4 | 7 | 12 | 20 | 33 |

Fibonacci numbers

What is $\sum_{i=0}^{i=n} F(i)$?

$P(n): \sum_{i=0}^{n} F(i) = F(n+2) - 1$

<u>Claim</u> $\forall n \in \mathbb{N}, \ P(n).$

<u>Base Case</u>

<u>Induction step</u> Assume n is an arbitrary natural number and that $P(n)$ is true.

Then $\sum_{i=0}^{n+1} F(i) = \left(\sum_{i=0}^{n} F(i) \right) + F(n+1)$

$= F(n+2) - 1 + F(n+1)$ (by IH)

$= \underbrace{F(n+3) - 1}_{\geq 2}, \text{ which } \underset{is}{} P(n+1)$ # since
$n+3 \geq 2$

Therefore $\forall n \in \mathbb{N}, \ P(n) \Rightarrow P(n+1)$
conclude $\forall n \in \mathbb{N}, P(n).$

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Notes