

CSC236 fall 2012

structural induction

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Using Introduction to the Theory of Computation, Chapter 4, Section 1.1



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Equivalence of inductions

Structural induction

Notes



$\underbrace{WO \Rightarrow MI \Rightarrow CI \Rightarrow WO}_{}$

The cycle is proved in the text, here is one link. Suppose you believe MI, and you have shown for some property P:

$$\forall n \in \mathbb{N}, (\forall 0 \leq i \land n, P(i)) \Rightarrow P(n)$$
 (1)

Now define a slightly different predicate:

 $\underbrace{P'(n): \forall 0 \leq i \leq n, P(i)}_{\text{including } n. \text{ Using only MI prove } \forall n, P'(n):$

Base case: Since we showed (1), and there are no natural numbers smaller than 0, we have P'(0).

Induction step: Assume n is an arbitrary natural number and that P'(n) is true. It follows from (1) that P(n+1) is true, and hence P'(n+1) is true. \checkmark

Define sets inductively

... so as to use induction on them later

One way to define the natural numbers:

$$\mathbb{N}$$
: The smallest set such that
1. $0 \in \mathbb{N}$
2. $n \in \mathbb{N} \Rightarrow n + 1 \in \mathbb{N}$.

 $\subseteq Q, \mathbb{R}, \mathbb{C},$

By smallest I mean \mathbb{N} has no proper subsets that satisfy these conditions. If I leave out smallest, what other sets satisfy the definition?

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What can you do with it?

The definition on the previous page defined the simplest natural number (0) and the rule to produce new natural numbers from old (add 1). Proof using Mathematical Induction work by showing that 0 has some property, and then that the rule to produce natural numbers preserves the property, that is $\begin{array}{c}
1. Prove P(0)
\end{array}$

2. Prove that
$$orall n \in \mathbb{N}, \ P(n) \Rightarrow P(n+1).$$
 $ightarrow$

Other structurally-defined sets

 $\begin{array}{c} \mathcal{E} \quad \text{includes} \quad & \chi, \, y, \, z \quad (\chi+g) \quad (y+z) \\ (y+\chi), \quad (y \times \chi) \quad ((\chi+g) \times (y+z)) \\ (\chi \times (y+z)) \\ \end{array}$ Define \mathcal{E} : The smallest set such that

$$\begin{array}{c} \blacktriangleright x,y,z \in \mathcal{E} \\ \flat e_1,e_2 \in \mathcal{E} \Rightarrow (e_1+e_2), \ (e_1-e_2), \ (e_1\times e_2), \\ \text{and} \ (e_1 \div e_2) \in \mathcal{E}. \end{array}$$

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Form some expressions in \mathcal{E} . Count the number of variables (symbols from $\{x, y, z\}$) and the number of operators (symbols from $\{+, \times, \div, -\}$). Make a conjecture.

Vr(e) = op(e) + 1

Structural induction $\left(\left(X + Y \right) + Z \right)$ P(e): vr(e) = op(e) + 1

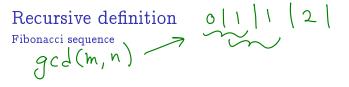
To prove that a property is true for all $e \in \mathcal{E}$, parallel the recursive set definition:

- ► Base case: Show that the property is true for the simplest members, {x, y, z}
- ► Induction step: Show "inheritance": if P(e₁) and P(e₂), then all possible combinations (e₁ + e₂), (e₁ e₂), (e₁ × e₂), and (e₁ ÷ e₂) have the property.

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Conclude that the property is true of all elements of \mathcal{E} .

Structural induction P(e): yr(e) = op(e) + 1 Conclude $\forall e \in \mathcal{E}, P(e), by SI$ $\# vo value \forall \# operations for the operations of the$ Prove $\forall e \in \mathcal{E}, P(e)$ induction). Proof (Structural induction). Base Case for basis elements $x, y, z \in \mathcal{E}$, each has, Vr(e) = 1 = 0 + 1 = op(e) + 1, so claim Induction step assume P(e,) and P(e2) hold for arbitrary CI, C, E E. Denote O e Et, -, x, -3, and note that Vr((e, Oez)) = Vr(ei) + Vr(ez). also note op((e, oez)) = $op(e_1) + op(e_2) + 1$. That means $op(le_1 \odot e_2)) = op(e_1) + op(e_2) + 1$ = $V(e_1) - 1 + V((e_2) - 1 + 1)$ (by 1H) = $Vr(e, S + Vr(e_2) - 1) = Vr((e, O, e_2)) - 1$ We have shown $P((E, O, e_2))$



This sequence comes up in applied rabbit breeding, the height of <u>AVL</u> trees, and the complexity of Euclid's algorithm for the GCD: $h^{-1} \rightarrow h^{-1} \sim 2$

$$F(n)=egin{cases} n \ f(n-2)+F(n-1) & n\geq 2 \ \end{array}$$

What is the sum of n Fibonacci numbers?

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What is $\sum_{i=0}^{i=n} F(i)$? P(n): $\sum_{i=0}^{n} F(i) = F(n+2) - 1$ Claim Yhe N, P(n), i=0 Base Case Induction step assume n'is an arbitrary natural monomber and that p(n) is true. Then $\sum_{i=0}^{n+1} F(i) = \left(\sum_{i=0}^{h} F(i)\right) + F(n+1)$ (by IH) = F(n+2) - 1 + F(n+1)= F(n+3) - 1, which # p(n+1) ## 50000 # N+322 There fore I ne IN, P(n) >> Pinei) conclude Vne Ni P(n). ・ロト ・ 雪 ト ・ ヨ ト 3

Notes

