CSC236 fall 2012

subtle induction

Danny Heap
heap@cs.toronto.edu
BA4270 (behind elevators)
http://www.cdf.toronto.edu/~heap/236/F12/
416-978-5899

Using Introduction to the Theory of Computation, Section 1.2–1.3

Outline

Well-ordering

Higher, and more, base cases

Well-ordering example

$\forall n, m \in \mathbb{N}, n \neq 0,\ R = \{r \in \mathbb{N} \mid \exists q \in \mathbb{N}, m = qn + r\}$ has a smallest element

This is the main part of proving the existence of a unique quotient and remainder:

$$\forall m \in \mathbb{N}, \forall n \in \mathbb{N} - \{0\}, \exists q, r \in \mathbb{N}, m = qn + r \wedge 0 \leq r < n$$

The course notes use Mathematical Induction. Well-ordering is shorter and clearer.

Principle of well-ordering

Every non-empty subset of $\mathbb{N}$ has a smallest element

$(0, 1)$

$\{ \frac{1}{n} \mid n \in \mathbb{N} - \{0\} \}$

Is there something similar for $\mathbb{Q}$ or $\mathbb{R}$?

For a given pair of natural numbers $m, n \neq 0$ does the set R satisfy the conditions for well-ordering?

$$R = \{ r \in \mathbb{N} \mid \exists q \in \mathbb{N}, m = qn + r \}$$

If so, we still need to be sure that

1. $0 \leq r < n$
2. That q and r are unique — no other natural numbers would work

...in order to have

$$\forall m \in \mathbb{N}, \forall n \in \mathbb{N} - \{0\}, \exists q, r \in \mathbb{N}, m = qn + r \wedge 0 \leq r < n$$

Every non-empty subset of $\mathbb{N}$ has a smallest element

$\forall m \in \mathbb{N}, \forall n \in \mathbb{N} - \{0\}, \exists q, r \in \mathbb{N}, m = qn + r \wedge 0 \leq r < n$: $P(m,n)$

Proof (well-ordering) that $\forall m \in \mathbb{N}, \forall n \in \mathbb{N}^+, P(m,n)$

Assume $m \in \mathbb{N}$ and $n \in \mathbb{N}^+$.

Then $R = \{r \in \mathbb{N} \mid \exists q \in \mathbb{N}, m = qn + r\}$ is non-empty, because $m \in R$, since $m = 0 \cdot n + m$. Then, by well ordering R has a least element, call it r'.

By membership in R, there must be some $q' \in \mathbb{N}$ s.t. $m = q'n + r'$. Also, since $R \subseteq \mathbb{N}$, $r' \geq 0$.

Suppose $r' \geq n$. But then $r' - n \in \mathbb{N}$ and $\in \mathbb{N}$

$$m = q'n + r' = q'n + r' - n + n = (q'+1)n + r' - n.$$

But then $r' - n \in R$, contradicting r' being least.

So $n > r' \geq 0$.

Then $\forall m \in \mathbb{N}, \forall n \in \mathbb{N}, P(m,n)$ -

Every non-empty subset of $\mathbb{N}$ has a smallest element

$\forall m \in \mathbb{N}, \forall n \in \mathbb{N} - \{0\}, \exists q, r \in \mathbb{N}, m = qn + r \wedge 0 \le r < n$

To show that r', q' are unique (read notes)

Suppose not, ie $r'', q'' \in \mathbb{N}$ and

$m = nq' + r' = nq'' + r''$

So $\underbrace{|q' - q''|}_{\ge 1} n = \underbrace{|r'' - r'|}_{\ge n}$, but $r', r'' \in [0, n-1]$

suppose

$\rightarrow\leftarrow$

Every non-empty subset of $\mathbb{N}$ has a smallest element

$P(n)$: Every round-robin tournament with n players that has a cycle has a 3-cycle

rules each player plays each diff player exactly once and there is only W V L

Claim: $\forall n \in \mathbb{N} - \{0, 1, 2\}, P(n)$.

Wins over

If there is a cycle $p_1 > p_2 > p_3 \ldots > p_n > p_1$, can you find a shorter one?

What happens in p_1 versus p_3

$p_1 > p_3$

$p_3 > p_1$

Every non-empty subset of $\mathbb{N}$ has a smallest element

$P(n)$: Every round-robin tournament with n players that has a cycle has a 3-cycle

Claim: $\forall n \in \mathbb{N} - \{0, 1, 2\}, P(n)$.

Proof (using WO)
assume $n \in \mathbb{N} - \{0,1,2\}$, and there is an n-tournament with a cycle.
Let $C = \{c \in \mathbb{N} - \{0,1,2\} \mid$ there is a c-cycle$\}$,
and $|C| > 0$ by assumption (there is a cycle)
So, by PWO there is some $c' \in C$ that the smallest. We claim that $c' = 3$. Suppose not, that there is a cycle $P_1 > P_2 > P_3 > \ldots > P_{c'} > P_1$.
There are 2 possibilities.
Case 1 $P_1 > P_3 \rightarrow$ show this lead to contradiction
Case 2 $P_3 > P_1 \rightarrow$ " "

Every non-empty subset of $\mathbb{N}$ has a smallest element

$P(n)$: Every round-robin tournament with n players that has a cycle has a 3-cycle

$2^n > 10n$

Where do we start?

P(n)

assume $n \in \mathbb{N} - \{0,1,2,3,4,5,?\}$ and that $2^n > 10 \cdot n$

Then $2^{n+1} = 2 \cdot 2^n$

$> 20n$

$> 10(n+1) = 10n + 10$

assumptions about n.

It's not true for several low values of n. You could re-write the predicate as $P'(n) : 2^{n+6} > 10(n+6)$, but why not just start later?

$3^n \geq n^3$: $P(n)$

Check your induction step

$P(0)$ ✓

$P(1)$ ✓

$P(2)$ ✓

$P(3)$ ✓

$P(4)$ ✓

Look at the graph.

$3^n \geq n^3$

Check your induction step

Base Cases. 0, 1, 2, 3

IH

Assume $n \in \mathbb{N}$, and that $\underbrace{3^n \geq n^3}$.

Then $3^{n+1} = 3 \cdot 3^n$

$\geq 3 \cdot n^3$

$= n^3 + \underbrace{n^3 + n^3}_{*}$

$\geq n^3 + 3n^2 + 3n + 1$

$\therefore = (n+1)^3$

$\boxed{n \geq 3}$

$n \geq 2$

Look at the graph.

$n^3 \geq 3n + 1$

$n^3 - 3n \geq 1$

$n(n^2 - 3) \geq 1$

$3^n \geq n^3$

Check your induction step

Look at the graph.

$3^n \geq n^3$

Check your induction step

Look at the graph.

$3^n \geq n^3$

Check your induction step

Look at the graph.