CSC236 fall 2012

subtle induction

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Using Introduction to the Theory of Computation, Section 1.2–1.3

Outline

Well-ordering

Higher, and more, base cases

Well-ordering example

$\forall n, m \in \mathbb{N}, n \neq 0,\ R = \{r \in \mathbb{N} \mid \exists q \in \mathbb{N}, m = qn + r\}$ has a smallest element

This is the main part of proving the existence of a unique quotient and remainder:

$$\forall m \in \mathbb{N}, \forall n \in \mathbb{N} - \{0\}, \exists q, r \in \mathbb{N}, m = qn + r \wedge 0 \leq r < n$$

The course notes use Mathematical Induction. Well-ordering is shorter and clearer.

Principle of well-ordering

Every non-empty subset of $\mathbb{N}$ has a smallest element

Is there something similar for $\mathbb{Q}$ or $\mathbb{R}$?

For a given pair of natural numbers $m, n \neq 0$ does the set R satisfy the conditions for well-ordering?

$$R = \{r \in \mathbb{N} \mid \exists q \in \mathbb{N}, m = qn + r\}$$

If so, we still need to be sure that

1. $0 \leq r < n$
2. That q and r are unique — no other natural numbers would work

. . . in order to have

$$\forall m \in \mathbb{N}, \forall n \in \mathbb{N} - \{0\}, \exists q, r \in \mathbb{N}, m = qn + r \wedge 0 \leq r < n$$

Every non-empty subset of $\mathbb{N}$ has a smallest element

$\forall m \in \mathbb{N}, \forall n \in \mathbb{N} - \{0\}, \exists q, r \in \mathbb{N}, m = qn + r \wedge 0 \leq r < n$

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Every non-empty subset of $\mathbb{N}$ has a smallest element

$P(n)$: Every round-robin tournament with n players that has a cycle has a 3-cycle

Claim: $\forall n \in \mathbb{N} - \{0, 1, 2\}, P(n)$.

If there is a cycle $p_1 > p_2 > p_3 \ldots > p_n > p_1$, can you find a shorter one?

Every non-empty subset of $\mathbb{N}$ has a smallest element

$P(n)$: Every round-robin tournament with n players that has a cycle has a 3-cycle

Claim: $\forall n \in \mathbb{N} - \{0, 1, 2\}, P(n)$.

Every non-empty subset of $\mathbb{N}$ has a smallest element

$P(n)$: Every round-robin tournament with n players that has a cycle has a 3-cycle

$2^n > 10n$: $P(n)$

Where do we start?

$P(0)$ ✓ $P(1)$ ✗ $P(2)$ ✗ $P(3)$ ✗ $P(4)$ ✗ $P(5)$ ✗ $P(6)$ ✓ $P(7)$ ✓ $P(8)$ ✓

Base?

It's not true for several low values of n. You could re-write the predicate as $P'(n) : 2^{n+6} > 10(n+6)$, but why not just start later?

omit Base case (for now) ← start at 6.

assume $n \in \mathbb{N} - \{0,1,2,3,4,5,?\}$, and that $P(n)$ is true, $2^n > 10n$ (IH)

Then, $2^{n+1} = 2 \cdot 2^n > 20n$ # (by IH)

$= 10n + 10n$

$\geq 10n + 10$. # since $n \geq 6 \geq 1$

$= 10(n+1)$

So, $P(n+1)$ follows.

So $\forall n \in \mathbb{N} - \{0,1,2,3,4,5\}$, $P(n) \Rightarrow P(n+1)$

Conclude $\forall n \in \mathbb{N} - \{0,1,2,3,4,5\}$, $P(n)$

$2^n > 10n$

Where do we start?

$3^n \geq n^3$: $P(n)$

Check your induction step

$P(0)$ ✓ $P(4)$ ✓

$P(1)$ ✓ $P(5)$ ✓

$P(2)$ ✓ $P(6)$ ✓

$P(3)$ ✓

Look at the graph.

$3^n \geq n^3$: P(n)

Check your induction step

Omit base(s) → 0, 1, 2, 3 Verify these separately, ie they are base cases.

Assume $n \in \mathbb{N} - \{0,1,2\}$ and that P(n) is true, that is $3^n \geq n^3$.

Then $3^{n+1} = 3 \cdot 3^n \geq 3n^3$]

$= n^3 + n^3 + n^3$

$\geq$

$n^3 + 3n^2 + 3n + 1$ # since $n^3 = n^3$ # and $n^3 \geq 3n^2$ # and $n^3 \geq$ # $3n+1$ # when $n \geq 3$

$= (n+1)^3$

$3n^3 \geq (n+1)^3$

0 ✗

1 ✗

1 2 ✗

1 1

1 2 1

1 3 3 1

Notes