CSC236 fall 2012

complete induction

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Using Introduction to the Theory of Computation, Section 1.3

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Outline

Principle of complete induction

Examples of complete induction

Complete Induction

another flavour needed

Every natural number greater than 1 has a prime factorization

$2 = 2$

$3 = 3$

$4 = 2 \times 2$

$5 = 5$

$6 = 2 \times 3$

$7 = 7$

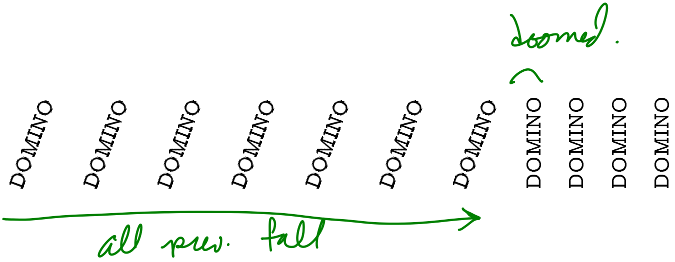
$8 = 2 \times 2 \times 2$

$9 = 3 \times 3$

Try some examples

How does the factorization of 8 help with the factorization of 9?

More dominoes



$$(\forall n \in \mathbb{N}, \langle P(0), \ldots, P(n-1)\rangle \Rightarrow P(n)) \Rightarrow \forall n \in \mathbb{N}, P(n)$$

If all the previous cases always implies the current case
then all cases are true

Every natural number greater than 1 has a prime factorization

$(\forall n \in \mathbb{N}-\{0,1\}, \langle P(\cancel{0}2), \ldots, P(n-1)\rangle \Rightarrow P(n))) \Rightarrow \forall n \in \mathbb{N}, P(n)$

Assume $n \in \mathbb{N} - \{0,1\}$, and that $P(j)$ is true for all natural numbers $1 < j < n$.

Case 1 n is prime. Then $n = n$ is its own prime factorization.

Case 2, n is not prime. Then n has at least 3 divisors in $\mathbb{N}$, 1, n, and $x \notin \{1, n\}$. Then $1 < x < n$. also $y = n/x$ is $\in \mathbb{N}$ and $1 < y < n$. (why?). So x, y are $\in \mathbb{N}$ $1 < x, y < n$, so $P(x)$ and $P(y)$ are true, by assumption (IH), so they have prime factorizations, that

Every natural number greater than 1 has a prime factorization

$(\forall n \in \mathbb{N}, \langle P(0), \ldots, P(n-1)\rangle \Rightarrow P(n)) \Rightarrow \forall n \in \mathbb{N}, P(n)$

$x = p_{x_1} \times \cdots \times p_{x_k}$, $y = p_{y_1} \times \cdots \times p_{y_i}$

So $n = xy = p_{x_1} \times \cdots \times p_{x_k} \times p_{y_1} \times \cdots \times p_{y_i}$, so

n has a prime factorization.

Since, in both possible cases, n has a prime factorization, $P(n)$ follows.

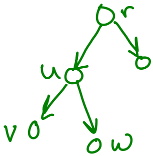
So, $\forall n \in \mathbb{N} - \{0, 1\}$, if $P(j)$ holds for every $j \in \mathbb{N}$ $1 < j < n$, then $P(n)$ follows.

$\forall n \in \mathbb{N} - \{0, 1\}$, $P(n)$ by CI

Conclude

Trees

definitions, page 32



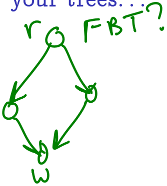
- A tree is a directed graph
- A non-empty tree has a root node, r, such that there is exactly one path from r to any other node.
- If a tree has an edge (u, v), then u is v's parent, v is u's child.
- Two nodes with the same parent are called siblings.
- A node with no children is called a leaf. A non-leaf is called an internal node.
- Binary trees have nodes with ≤ 2 children, and children are labelled either left or right.

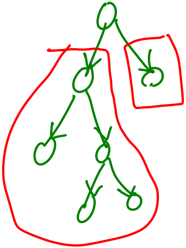
FBT

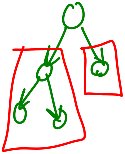
- Internal nodes of full binary trees have 2 children.

Tree examples

know your trees. . .







Every full binary tree, except the zero tree, has an odd number of nodes

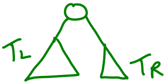
$(\forall n \in \mathbb{N}, \langle P(0), \ldots, P(n-1)\rangle \Rightarrow P(n)) \Rightarrow \forall n \in \mathbb{N}, P(n)$

Assume n is a natural number greater than 0, and that if T is a FBT with i nodes, and $0 < i < n$, then i is odd. Must show that if there is a FBT with n nodes, then n is odd.

Suppose T is a FBT with n nodes.

Case 1 $n = 1$. If there is a FBT with $n = 1$ node, then $n = 1$ is odd

case 2, $n > 1$ Then the root has exactly 2 children and they are roots of FBT (non-empty). call them T_L and T_R



, with j_L nodes in T_L and j_R nodes in T_R.

Every full binary tree, except the zero tree, has an odd number of nodes

$(\forall n \in \mathbb{N}, \langle P(0), \ldots, P(n-1)\rangle \Rightarrow P(n)) \Rightarrow \forall n \in \mathbb{N}, P(n)$.

→ Since T_L is non empty $j_L > 0$, similary $j_R > 0$
also, since j_L and j_R are pos. number that add to $n-1$, $j_L, j_R < n$, so IH applies
and j_L, j_R are odd so $n = j_L + j_R + 1$
is odd.
Since claim hold in both possible cases, every non-empty FBT ~~ha~~ with n nodes has n odd.
Since assumed only $n \in \mathbb{N} - \{0\}$, and if T is FBT with i nodes, $0 < i < n$, and then derived $P(n)$, this shows $\forall n \in \mathbb{N} - \{0\}$, $P(i)\ \forall\, 0 < i < n \Rightarrow P(n)$.

Conclude: $\forall n \in \mathbb{N} - \{0\}, P(n)$.

Every rectangular array of chocolate $m \times n$ squares can be broken up with ? "breaks"

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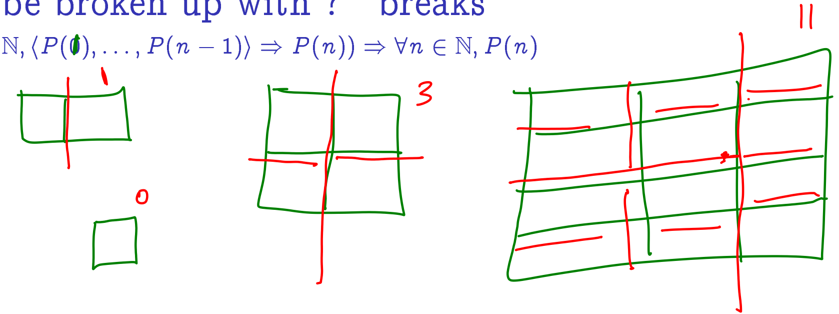

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Every rectangular array of chocolate $m \times n$ squares can be broken up with ? "breaks"

$(\forall n \in \mathbb{N}, \langle P(0), \ldots, P(n-1)\rangle \Rightarrow P(n)) \Rightarrow \forall n \in \mathbb{N}, P(n)$

After a certain natural number n, every postage can be made up by combining 3− and 5− cent stamps

0 ✓

1 ✗

2 ✗

3 ✓

4 ✗

5 ✓

6 ✓

$\boxed{7 ✗}$

8 ✓

9 ✓

10 ✓

11 ✓

12 ✓

13 ✓

14 ✓

15 ✓

16

⋮

Base Case 8 at least

9, 10?

Suppose we have postage for n, and it includes at least 1 5¢ stamp.

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After a certain natural number n, every postage can be made up by combining $3-$ and $5-$ cent stamps

option 1

Single base case $P(8)$, but two cases for $n = 5 +$ something or $n = 3 \times$ something.

option 2

either 3 or 5 (!) base cases, argue by ~~on~~ complete induction that you can add $i + 3 = n$, if $n - 3 \geq 8$ or $j + 5 = n$, if $n - 5 \geq 8$