CSC236, Fall 2012
Assignment 2
Sample solution

1. Odd Maximal Contiguous Ones Free Strings (OMCOFS) are binary strings that contain no maximal contiguous substring[1]of 1s that is of odd length. For example 0110 is an OMCOFS because the only maximal substring of 1s it contains is 11, and that is not of odd length. On the other hand 10111 is not an OMCOFS because it contains 1 and 111 — both maximal contiguous substrings of 1s, and both of odd length.

   Define $H(n)$ as the number of OMCOFS in the set of binary strings of length n. For example, $H(4) = 5$, since we have the following binary strings (and no others) of length 4 that are free of maximal contiguous substrings of odd length:

   0000 1100 0110 0011 1111

   Develop a recurrence (a recursive definition) for $H(n)$, and explain why it correctly counts the number of OMCOFS of length n. Unwind (use repeated substitution) your recurrence and find a closed form for $H(n)$. Prove that your closed form is equal to the recursive definition of $H(n)$ using an appropriate flavour of induction.

   Solution: There is exactly one binary string of length 0, the empty string, which contains no maximal contiguous substring of 1s of odd length, so $H(0) = 1$. The only binary string of length 1 that contains no maximal contiguous substrings of odd length is the string 0, so $H(1) = 1$. For $n > 1$, the $OMCOFSs$ of length n can be partitioned into those that end in 0 and those that end in 1. Those that end in 0 are in 1–1 correspondence with the OMCOFSs of length $n - 1$ (by appending or removing a 0). An OMCOFS of length n that ends in 1 certainly can't end in 01, so these must end in 11. Thus, the OMCOFSs of length n that end in 1 are in 1–1 correspondence with the OMCOFSs of length $n - 2$ (by appending or removing 11). So the following recurrence counts the OMCOFSs of length n:

$$H(n) = \begin{cases} 1 & \text{if } n < 2 \\ H(n-2) + H(n-1) & \text{if } n > 1 \end{cases}$$

   The form of the recurrence makes it seem likely that H is Fibonacci-esque, and I can firm up this conjecture by repeated substitution. Assuming that n is sufficiently greater than 1 to apply the

[1] Maximal in the sense that every larger substring that contains them has at least one 0

second branch of the recurrence a few times, I explore the pattern:

$$\begin{aligned}
H(n) &= H(n-1)+H(n-2) = H(n-3)+H(n-2)+H(n-2) \\
&= 2H(n-2)+H(n-3) = 2\left(H(n-4)+H(n-3)\right)+H(n-3) \\
&= 3H(n-3)+2H(n-4) = 3\left(H(n-5)+H(n-4)\right)+2H(n-4) \\
&= 5H(n-4)+3H(n-5) \\
&\vdots \\
&= F(i)H(n-i+1)+F(i-1)H(n-i) \\
&\qquad \#\quad \text{Fib(i)} \equiv F(i) \qquad 1 \le i \le n \\
&\vdots \\
&= F(n)H(1)+F(n-1)H(0) \qquad \#\quad i=n \\
&= F(n)+F(n-1) = F(n+1)
\end{aligned}$$

I know a closed form for $F(n+1)$, so now I set out to prove:

Claim: $H(n) = (\phi^{n+1} - \hat{\phi}^{n+1})/\sqrt{5}$, where $\phi = \frac{1+\sqrt{5}}{2}$ and $\hat{\phi} = \frac{1-\sqrt{5}}{2}$ are the solutions to the equation $x^2 = x+1$.

Proof, by complete induction on n:

Induction step: Assume $n \in \mathbb{N}$ and that $H(k) = (\phi^{k+1} - \hat{\phi}^{k+1})/\sqrt{5}$ for every natural number k, $0 \le k < n$.

Case $n < 2$: In these cases:

$$\begin{aligned}
H(0) &= 1 = \frac{1+\sqrt{5}-1+\sqrt{5}}{2\sqrt{5}} = \frac{\phi^1 - \hat{\phi}^1}{\sqrt{5}} \\
H(1) &= 1 = \frac{1+\sqrt{5}+2-1+\sqrt{5}-2}{2\sqrt{5}} = \frac{(\phi+1)-(\hat{\phi}+1)}{\sqrt{5}} = \frac{\phi^2 - \hat{\phi}^2}{\sqrt{5}} \\
&= \qquad \#\quad \phi^2 = \phi+1 \quad \text{and} \quad \hat{\phi}^2 = \hat{\phi}+1
\end{aligned}$$

So the claim is verified for $H(0)$ and $H(1)$ directly.

Case $n > 1$: In this case:

$$\begin{aligned}
H(n) &= H(n-1)+H(n-2) \qquad \#\text{by definition of } H(n), n > 1 \\
&= \frac{\phi^n - \hat{\phi}^n}{\sqrt{5}} + \frac{\phi^{n-1} - \hat{\phi}^{n-1}}{\sqrt{5}} \qquad \#\quad \text{by assumption, } 0 \le n-2, n-1 < n \\
&= \frac{(\phi^n + \phi^{n-1}) - (\hat{\phi}^n + \hat{\phi}^{n-1})}{\sqrt{5}} = \frac{\phi^{n-1}(\phi+1) - \hat{\phi}^{n-1}(\hat{\phi}+1)}{\sqrt{5}} \\
&= \frac{\phi^{n-1}\phi^2 - \hat{\phi}^{n-1}\hat{\phi}^2}{\sqrt{5}} = \frac{\phi^{n+1} - \hat{\phi}^{n+1}}{\sqrt{5}} \\
&= \qquad \#\quad \phi^2 = \phi+1 \quad \text{and} \quad \hat{\phi}^2 = \hat{\phi}+1
\end{aligned}$$

So, the claim holds in this case also.

Thus, if the claim is true of $H(k)$ for all natural numbers $0 \le k < n$, it is also true of $H(n)$. I conclude that $\forall n \in \mathbb{N}, H(n) = \frac{\phi^{n+1} - \hat{\phi}^{n+1}}{\sqrt{5}}$, by complete induction.

2. In class we developed the following recurrence to express the time complexity of binary search:

$$T(n) = \begin{cases} 1 & \text{if } n = 1 \\ 1 + \max\{T(\lceil n/2 \rceil), T(\lfloor n/2 \rfloor)\} & \text{if } n > 1 \end{cases}$$

You have already found a closed form for $T(n)$ when n is a power of 2, using unwinding (repeated substitution). Now emulate the proof from the Course Notes, Lemma 3.6, page 84 that the recurrence for MergeSort is non-decreasing, to prove that our T is also non-decreasing. Use this fact, and no further induction, to prove that $T \in \theta(\lg n)$.

Solution: The first step is to show that T is nondecreasing. I define a predicate, slightly modified from Lemma 3.6 ($m \leq n$, rather than $m < n$)

$$P(n) : \forall m \in \mathbb{N}, m \leq n \Rightarrow T(m) \leq T(n)$$

I claim $\forall n \in \mathbb{N}^+, P(n)$

Proof, by complete induction:

Induction step: Assume that n is an arbitrary positive natural number, and that for every natural number $1 \leq k < n$ $P(n)$ is true. I consider cases where $n < 3$ and $n > 2$.

Case $1 \leq n < 3$: $P(1)$ is vacuously true, since there are no positive integers less than 1. To test $P(2)$ I have to evaluate:

$$T(1) = 1 \leq T(2) = 1 + 1 = 2 \qquad \# \quad \text{by definition of } T$$

In both cases, $P(n)$ holds.

Case $n > 2$: By Lemma 3.3 and $n > 1$ I know that $1 \leq \lfloor n/2 \rfloor \leq \lfloor n/2 \rfloor$ and $1 \leq n - 1 < n$, so by assumption $P(\lfloor n/2 \rfloor)$, $P(\rceil n/2 \rceil)$, and $P(n-1)$ all hold. Also, $1 \leq \lceil (n-1)/2 \rceil, \lfloor (n-1)/2 \rfloor$, since $n \geq 3$. And, since $P(n-1)$ is assumed, by transitivity of $\leq$ I need only show that $T(n) \geq T(n-1)$. I have

$$\begin{aligned} T(n-1) &= 1 + \max\{T(\lceil (n-1)/2 \rceil), T(\lfloor (n-1)/2 \rfloor)\} \qquad \# \quad \text{by IH for } n-1 \\ &\leq 1 + \max\{T(\lceil n/2 \rceil, T(\lfloor n/2 \rfloor)\} \\ &\qquad\qquad \# \quad \text{by IH for } \lceil n/2 \rceil, \lfloor n/2 \rfloor \quad \text{and nondecreasing property of floor, ceiling} \\ &= T(n) \end{aligned}$$

So $P(n)$ holds in this case.

So, $\forall n \in \mathbb{N}^+$, if $P(k)$ holds for $1 \leq k < n$, then $P(n)$ holds also.

I conclude that $\forall n \in \mathbb{N}, P(n)$, by complete induction.

Prove that $T \in \theta(\lg n)$: I assume, without proof (done in class) that if n is a power of 2, then $T(n) = 1 + \lg n$. For convenience, I define $\hat{n} = 2^{\lceil \lg n \rceil}$, the "next" power of 2 no smaller than n. Since $\lceil x \rceil < x + 1$, I know that $\hat{n}/2 < n \leq \hat{n}$, and both $\hat{n}$ and $\hat{n}/2$ are powers of 2, provided $n \geq 2$.. I use these to prove:

$$\exists c_1, c_2 \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow c_1 \lg n \leq T(n) \leq c_2 \lg n$$

In other words, I prove that $T \in \theta(\lg n)$.

Pick $c_1 = 1/2$, $c_2 = 3$, and $B = 4$. Then $c - 1, c_2 \in \mathbb{R}^1$ and $B \in \mathbb{N}$. Assume $n \geq B$, then

$$\begin{aligned} T(n) &\leq T(\hat{n}) \quad \# \quad T \text{ nondecreasing and } n \leq \hat{n} \\ &= 1 + \lg \hat{n} \quad \# \quad \hat{n} \text{ is a power of } 2, \lg \text{ is nondecreasing} \\ &< 1 + \lg(2n) \quad \# \quad \hat{n}/2 < n \Rightarrow \hat{n} < 2n \\ &= 1 + \lg n + \lg 2 = \lg 2 + 2 \leq \lg n + 2 \lg n \quad \# \quad n \geq B \geq 2 \\ &= 3 \lg n = c \lg n \end{aligned}$$

. . . also

$$\begin{aligned} T(n) &\geq T(\hat{n}/2) \quad \# \quad T \text{ nondecreasing and } n > \hat{n}/2 \\ &= 1 + \lg \hat{n}/2 \quad \# \quad \hat{n}/2 \text{ is a power of } 2 \\ &\geq 1 + \lg n/4 \quad \# \quad \hat{n} > n/2 \\ &= 1 + \lg n - \lg 4 = \lg n - 1 = \frac{1}{2} \lg n + \frac{1}{2} \lg n - 1 \\ &\geq \frac{1}{2} \lg n \quad \# \quad n \geq B = 4 \Rightarrow \lg n \geq 2 \\ &= c_1 \lg n \end{aligned}$$

Thus,

$$\exists c_1, c_2 \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow c_1 \lg n \leq T(n) \leq c_2 \lg n$$

since $c_1 = 1/2$, $c_2 = 3$, and $B = 4$ satisfy the claim.