CSC236, Fall 2012
Assignment 1

These problems aim to give you some practice writing proofs of facts from different domains, using induction. Unless you find them easy, you should start working on them early, and be sure to talk them over with your instructor and teaching assistant.

Submit your solutions as a PDF file called `a1.pdf`. You must generate the PDF from a word processor, or LaTeX — no scanned handwritten work will be accepted.

1. A **ternary tree** is a tree where each node has no more than 3 children, and the **height** of a tree is defined as the number of nodes in the longest path[1] from the root to any leaf. Use Complete Induction to prove that if a[2] ternary tree has height n, it has no more than $3^n - 2$ nodes.

   Proof (using Complete Induction): Assume that n is a positive natural number, and that for every $1 \leq i < n$ $P(i)$ is true, that is every ternary tree of height i has no more than $3^i - 2$ nodes. I must show that $P(n)$ follows

   Case 1, $n = 1$: This tree consists of only the root, and has $1 \leq 3^1 - 2 = 1$ nodes. This verifies $P(1)$.

   Case 2, $n > 1$: The root of this tree has at most three non-empty subtrees. Without loss of generality, I will assume three non-empty subtrees (since the presence of a non-empty subtree cannot decrease the numer of nodes) with heights k_1, k_2, and k_3. Notice that $1 \leq k_1, k_2, k_3 < n$, since the longest path from root to leaf in a subtree must be strictly less than the corresponding path that includes the original root. This means we may assume $P(k_1)$, $P(k_2)$, and $P(k_3)$. Thus the tree has no more (including the root) than $1 + 3^{k_1} - 2 + 3^{k_2} - 2 + 3^{k_2} - 2$ nodes, by the induction hypothesis. Since the $k_i \leq n - 1$, this implies the tree has no more than $3 \times 3^{n-1} - 6 + 1$ or $3^n - 5$ nodes.

   Since in both possible cases there are no more than $3^n - 2$ nodes, I have established $P(n)$.

   So, $\forall n \in \mathbb{N} - \{0\}$, if $P(i)$ is true for every $1 \leq i < n$, then $P(n)$ follows.

   I conclude that for every positive natural number n, $P(n)$ is true.

2. Exponential growth is, typically, faster than polynomial growth, so you should expect that beyond the first few natural numbers $n^3 + n < 3^n$. Determine what the "first few" natural numbers are, and prove that this inequality holds for all other natural numbers, using Mathematical Induction.

   $P(n) : n^3 + n < 3^n$. The claim is false for $n \in \{2, 3\}$. I'll prove that $\forall n \in \mathbb{N} - \{0, 1, 2, 3\}$, $P(n)$.

   Proof (by Mathematical Induction)

[1] Some mathematicians define tree **height** to be the number of edges in a longest path.
[2] non-empty — the claim is false for the empty tree

Base case, $n = 4$: In this case $4^3 + 4 = 68 < 81 = 3^4$, so $P(n)$ is true for $n = 4$.

Induction step: Assume that n is a generic natural number greater than 3, and that $P(n)$ is true. I must show that $P(n+1)$ follows.

$$\begin{aligned}
3^{n+1} = 3 \times 3^n &> 3(n^3 + n) \quad \# \quad \text{by the induction hypothesis} \\
&= n^3 + n^3 + n^3 + 3n \geq n^3 + n^3 + 2n + 3n \quad \# \quad n > 2 \Rightarrow n^2 > 2. \\
&> n^3 + 3n^2 + 4n + 2 \quad \# \quad n > 3 \Rightarrow n^3 > 3n^2, n > 2 \\
&= (n+1)^3 + (n+1)
\end{aligned}$$

So $3^{n+1} > (n+1)^3 + (n+1)$, that is $P(n)$.

Since n is assumed to be a typical natural number greater than 3, $\forall n \in \mathbb{N} - \{0,1,2,3\}$, $P(n) \Rightarrow P(n+1)$.

I conclude that $\forall n \in \mathbb{N} - \{0,1,2,3\}$, $P(n)$, by Mathematical Induction.

3. A set of 3 elements has exactly one subset of size 3 (I'll call it a 3-subset), namely itself. Experiment until you find a formula for the number of 3-subsets that a set of $n+3$ elements has, then use Mathematical Induction to prove that your formula works for any natural number n. You may use, without proof, that a set with $n+2$ elements has $[(n+2)(n+1)]/2$ subsets of size 2.

   $P(n)$ Every set with $n+3$ elements has $(n+3)(n+2)(n+1)/6$ three-subsets. I will prove that $P(n)$ is true for every natural number n.

   Proof (using Mathematical Induction): Assume that n is an arbitrary natural number, and that $P(n)$ is true. I must show that $P(n+1)$ follows, that is every set with $(n+1)+3$ elements has $(n+4)(n+3)(n+2)/6$ 3-subsets.

   Let S be an arbitrary set with $|S| = n+1$ elements. Since n is a natural number, $n+1 > 0$ and S is not empty, so I can distinguish one of its elements and call it t. I can partition the 3-subsets of S into $\mathcal{T}_-$, the set of 3-subsets of S that don't contain t, and $\mathcal{T}_+$, the set of 3-subsets of S that do contain t.

   To count $\mathcal{T}_-$, notice that it consists of the 3-subsets of $S - t$, which has n elements, so (by the Induction Hypothesis) I have $|\mathcal{T}_-| = (n+3)(n+2)(n+1)/6$.

   To count $\mathcal{T}_+$, notice that it consists of 3-subsets of S that are in one-to-one correspondence with the 2-subsets of $S - \{t\}$, since they can be matched be adding (or removing) the element t. By assumption I know that $S - \{t\}$, a set with $(n+2)+1$ elements, has $(n+3)(n+2)/2$ 2-subsets, so $|\mathcal{T}_+| = (n+3)(n+2)/2$.

   Adding the two portions of the partition, the total number of 3-subsets of S is

   $$\begin{aligned}
   \frac{(n+3)(n+2)(n+1)}{6} + \frac{(n+3)(n+2)}{2} &= \frac{(n+3)(n+2)(n+1) + 3(n+3)(n+2)}{6} \\
   &= \frac{(n+3)(n+2)(n+1+3)}{6} = \frac{(n+4)(n+3)(n+2)}{6}
   \end{aligned}$$

   This verifies $P(n+1)$.

   So, $\forall n \in \mathbb{N}$, if every set of size $n+3$ has $(n+3)(n+2)(n+1)/6$ 3-subsets, then every set of size $n+1$ has $(n+4)(n+3)(n+2)/6$ 3-subsets.

   I conclude that $\forall n \in \mathbb{N}$, every set of size $n+3$ has $(n+3)(n+2)(n+1)/6$ 3-subsets, by Mathematical Induction.

4. Use Complete Induction or Mathematical Induction[3] to prove that any binary string that begins and ends with the same bit has an even number of occurrences of substrings from $\{01, 10\}$, e.g. 010 has two: 01 and 10. You may find it useful to combine this claim with a similar claim about binary strings that begin and end with different bits, and then prove the combined claims simultaneously.

   $P(n)$: A binary string of length n that begins and ends with the same bit has an even number of occurrences of substrings from $\{01, 10\}$, and a binary string of length n that begins and ends with different bits has an odd number of occurrences of substrings from $\{01, 10\}$. I will prove that for all natural numbers, $P(n)$ is true.

   The string of length 0 doesn't begin and end with the same or different bits, so any claim about zero-length strings with these properties is vacuously true, and unhelpful in induction. We'll work on the claim for strings of length at least 1.

   Proof (by Mathematical Induction):

   Base case, P(1): The only strings of length 1 are 0 and 1. Each of them begin and end with the same bit, and they each contain and even number (zero) of substrings from $\{01, 10\}$.

   Induction step: Assume that $n \in \mathbb{N} - \{0\}$ and that $P(n)$ is true. I have to show that $P(n+1)$ follows.

   Suppose s is a binary string of length $n+1$. Then s has two initial bits, and they are either the same or different.

   Case 1, the two initial bits of s are the same: The first two bits do not increase the number of substrings from $\{01, 10\}$, so these are determined by the last n bits. If s begins and ends with the same bits, so do its last n bits, and by the induction hypothesis the last n bits have an even number of occurrences of substrings from $\{01, 10\}$, so s does also. If s begins and ends with different bits, so do its last n bits, and by the induction hypothesis the last n bits of s have an odd number of occurrence of strings from $\{01, 10\}$, so s does also.

   Case 2, the two initial bits of s are different: The first two bits increase the number of substrings from $\{01, 10\}$ by one over those in the last n bits of s. If s begins and ends with the same bits, then the last n bits of s begin and end with different bits, so by $P(n)$ the last n bits have an odd number of occurrences of substrings from $\{01, 10\}$, so s has an even num ber. If s begins and ends with different bits, then its last n bits ends with the same bits, so by the induction hypothesis the last n bits have an even number of occurrences of substrings from $\{01, 10\}$ so s has an odd number.

   In both cases $P(n+1)$ follows.

   Since n was an arbitrary positive integer, I conclude that $\forall n \in \mathbb{N} - \{0\}$, $P(n) \Rightarrow P(n+1)$.

   I conclude that for all positive natural numbers n, $P(n)$, by Mathematical Induction.

[3] Several other proof techniques will also work