Office hours / W 2-4 BA4270
MTWR- 4-6 - Help Centre

Extra TA office hours announce RSN

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Mathematical expression

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Course notes, chapter 3, 4

Outline

inference rules

asymptotics

proof about limits

$\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+, \forall y \in \mathbb{R}, |y - \pi| < \delta \Rightarrow |y^2 - \pi^2| < \varepsilon$

Assume $\varepsilon \in \mathbb{R}^+$ # to introduce $\forall$

Pick $\delta = ?$ # to introduce $\exists$

Then $\delta \in \mathbb{R}^+$ # figure out why later

Assume $y \in \mathbb{R}$ # to introduce $\forall$

Assume $|y - \pi| < \delta$ # to introduce $\Rightarrow$

$\vdots$

Then $|y^2 - \pi^2| < \varepsilon$

Then $|y - \pi| < \delta \Rightarrow |y^2 - \pi^2| < \varepsilon$ # introduced $\forall$

Then $\forall y \in \mathbb{R}, |y - \pi| < \delta \Rightarrow |y^2 - \pi^2| < \varepsilon$ # introduced $\forall$

Then $\exists \delta \in \mathbb{R}^+, \forall y \in \mathbb{R}, |y - \pi| < \delta \Rightarrow |y^2 - \pi^2| < \varepsilon$
introduced $\exists$

Conclude $\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+, \forall y \in \mathbb{R}, |y - \pi| < \delta \Rightarrow |y^2 - \pi^2| < \varepsilon$

introduced $\forall$

fill in the $\vdots$

proof by cases

Sometimes your argument has to split to take into account possible properties of your generic element:

$$\forall n \in \mathbb{N}, n^2 + n \text{ is even}$$

A natural approach is to factor $n^2 + n$ as $n(n+1)$, and then consider the case where n is odd, then the case where n is even.

scratch

get wrong right

Be careful proving a claim false. Consider the claim, for some suitably defined X, Y and P, Q:

$$S: \qquad \forall x \in X, \forall y \in Y, P(x,y) \Rightarrow Q(x,y)$$

To **disprove** S, should you prove:

$$\forall x \in X, \forall y \in Y, P(x,y) \Rightarrow \neg Q(x,y)$$

What about

$$\forall x \in X, \forall y \in Y, \neg(P(x,y) \Rightarrow Q(x,y))$$

Explain why, or why not.

Define $T(n)$ by: Definition

$$\forall n \in \mathbb{N} \qquad T(n) \Leftrightarrow \exists i \in \mathbb{N}, n = 7i + 1.$$

Take some scrap paper, **don't** write your name on it, and fill in as much of the proof of the following claim as possible:

S1

$$\forall n \in \mathbb{N}, T(n) \Rightarrow T(n^2)$$

Now fill in as much of the **disproof** of the following claim as possible:

S2

$$\forall n \in \mathbb{N}, T(n^2) \Rightarrow T(n)$$

allowed inference

At this point you've been introduced to some rules of inference, that allow you to reach conclusions in certain situations. You may use these (see pages 44–46 of the course notes) to guide your thinking, or as marginal notes to justify certain steps:

conjunction elimination: If you know $A \wedge B$, you can conclude A separately (or B separately).

existential instantiation: If you know that $\exists k \in X, P(k)$, then you can certainly pick an element with that property, let $k' \in X, P(k')$. # Sometimes just use this symbol

disjunction elimination: If you know $A \vee B$, the additional information $\neg A$ allows you to conclude B.

implication elimination: If you know $A \Rightarrow B$, the additional information A allows you to conclude B. On the other hand, the additional information $\neg B$ allows you to conclude $\neg A$.

universal elimination: If you know $\forall x \in X, P(x)$, the additional information $a \in X$ allows you to conclude $P(a)$.

more inferences

$(\exists x \ldots) \Rightarrow (\quad)$

Here are some rules that allow you to introduce new logical structures

implication introduction: If you assume A and, under that assumption, B follows, than you can conclude $A \Rightarrow B$.

antecedent

consequent.

universal introduction: If you assume that a is a generic element of D and, under that assumption, derive $P(a)$, then you can conclude $\forall a \in D, P(a)$.

#intro $\forall$

existential introduction: If you show $x \in X$ and you show $P(x)$, then you can conclude $\exists x \in X, P(x)$.

conjunction introduction: If you know A and you know B, then you can conclude $A \wedge B$.

disjunction introduction: If you know A you can conclude $A \vee B$.

sorting strategies

Which algorithm do you use to sort a 5-card euchre hand?

- insertion sort
- selection sort
- some other sort?

If you use one of the first two, the number of "steps" you execute will more than quadruple if you graduate from euchre to a 13-card bridge hand.

If you were enough of a virtuoso to use mergesort or quicksort on your cards, the change from euchre to bridge would roughly double your work.

We are most interested in how quickly running-time grows with the size of the problem, since these quickly swamp constant-factor differences between algorithms that are of the same "order."

different, but the same?

Suppose you could count the "steps" required by an algorithm in some sort of platform-independent way. You would find that the steps required for insertion sort and selection sort on lists of size n were no more than some quadratic functions of n

To a computer scientists, even though they may vary by substantial constant factors, all quadratic functions are the "same" — they are in $\mathcal{O}(n^2)$.

$$g(n) = n^2 \qquad f(n) = 3n^2 + 50 \qquad h(n) = 15n^2 + n$$

notes

To Prove $\forall n \in \mathbb{N}, T(n) \Rightarrow T(n^2)$.

Assume $n \in \mathbb{N}$ # to intro $\forall$

Assume $T(n)$, ie $\exists i \in \mathbb{N}, n = \underbrace{7i+1}$ < Pick $i_0, n = 7i_0+1$ ...

Then $n^2 = (7i+1)^2$

$= 49i^2 + 14i + 1$

$= 7(7i^2 + 2i) + 1$

Then let $j = 7i^2 + 2i$

Then $n^2 = 7j + 1$

and $j \in \mathbb{N}$, since $j = 7 \cdot i \cdot i + 2 \cdot i$ & $7, i, 2 \in \mathbb{N}$.

Then $T(n^2)$.

Then $T(n) \Rightarrow T(n^2)$ # introduced $\Rightarrow$

Then $\forall n \in \mathbb{N}, T(n) \Rightarrow T(n^2)$ # introduced $\forall$.

notes

Prove $\neg(\forall n \in \mathbb{N}, T(n^2) \Rightarrow T(n))$

That $\exists n \in \mathbb{N}, T(n^2) \wedge \neg T(n)$

Let $n = 6$.

Then $n \in \mathbb{N}$ # $6 \in \mathbb{N}$, c'mon!

Then $n^2 = 36 = 7 \cdot 5 + 1$

Then $T(n^2)$

also $n = 7 \cdot 0 + 6$, so by uniqueness of remainder $\nexists i \in \mathbb{N}, n = 7i + 1$.

Then $\exists n \in \mathbb{N}, T(n^2) \wedge \neg T(n)$