A2 − due ≈ 10 days prove/disprove.

CSC165 winter 2013

Mathematical expression

diagonal

"solution"

1×3, 3×5, 5×7,

1×2, 2×3, 3×4, 4×5

9×15

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Course notes, chapter 3

Outline

notes

proof about limits

$\varepsilon \to e$, $\delta \to d$

$\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+, \forall y \in \mathbb{R}, |y - \pi| < \delta \Rightarrow |y^2 - \pi^2| < \varepsilon$

Assume $\varepsilon \in \mathbb{R}^+$ # to introduce $\forall$

Pick $\delta = ?$ ~~# to introduce $\exists$~~ → δ = what I said next page.

Then $\delta \in \mathbb{R}^+$ # figure out why later

Assume $y \in \mathbb{R}$ # to introduce $\forall$

Assume $|y - \pi| < \delta$ # to introduce $\Rightarrow$

$\vdots$

Then $|y^2 - \pi^2| < \varepsilon$

Then $|y - \pi| < \delta \Rightarrow |y^2 - \pi^2| < \varepsilon$ # introduced ~~$\forall$~~ $\Rightarrow$

Then $\forall y \in \mathbb{R}, |y - \pi| < \delta \Rightarrow |y^2 - \pi^2| < \varepsilon$ # introduced $\forall$

Then $\exists \delta \in \mathbb{R}^+, \forall y \in \mathbb{R}, |y - \pi| < \delta \Rightarrow |y^2 - \pi^2| < \varepsilon$
introduced $\exists$

Conclude $\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+, \forall y \in \mathbb{R}, |y - \pi| < \delta \Rightarrow |y^2 - \pi^2| < \varepsilon$
introduced $\forall$

fill in the :

Then $|y^2 - \pi^2| = |y-\pi| \cdot |y+\pi|$ # factor,

$< \delta\, |y+\pi|$ # $|y-\pi| < \delta$, assump

$= \delta\, |y \underbrace{-\pi+\pi} + \pi|$ # add 0

$\leq \delta\, ||y-\pi| + 2\pi|$ # $\triangle$-inequality

$|-1+5|$

$< \delta\, |\delta + 2\pi|$ # $\leq ||-1|+5|$

$\leq \delta\, |1 + 2\pi|$ # $\delta \leq 1$ (*)

$= \delta(1+2\pi)$ # 1, 2π +-pve

$\leq \varepsilon$ # $\delta \leq \frac{\varepsilon}{1+2\pi}$ (*)

$\delta = \min(1, \frac{\varepsilon}{1+2\pi})$ works.

proof by cases

Sometimes your argument has to split to take into account possible properties of your generic element:

$$\forall n \in \mathbb{N}, n^2 + n \text{ is even}$$

A natural approach is to factor $n^2 + n$ as $n(n+1)$, and then consider the case where n is odd, then the case where n is even.

scratch

assume $n \in \mathbb{N}$ # in order to introduce $\forall$

Then $(\exists k \in \mathbb{N}, n = 2k) \vee (\exists k \in \mathbb{N}, n = 2k+1)$

Case 1 assume $\exists k \in \mathbb{N}, n = 2k$

Then $n^2 + n = n(n+1)$ # factor.

$= 2k(2k+1)$ # sub $n = 2k$.

$= 2(k(2k+1))$ # factoring

Then $\exists k' \in \mathbb{N}, n^2+n = 2k'$

$k' = k(2k+1) \in \mathbb{N}$, since $2, 1, k \in \mathbb{N}$

Then n^2+n is even.

Case 2 assume $\exists k \in \mathbb{N}, n = 2k+1$

Then $n^2+n = n(n+1)$

$= (2k+1)(2k+1+1)$ # sub in

$= (2k+1)(2k+2)$

$= 2(2k+1)(k+1)$ # factor out 2

similar

Since n^2+n is even in both possible case, conclude true for n.

get wrong right

Be careful proving a claim false. Consider the claim, for some suitably defined X, Y and P, Q: $X = Y = \{1\}$

✓

$$S: \quad \underbrace{\forall x \in X, \forall y \in Y, P(x,y) \Rightarrow Q(x,y)}_{\exists x \in X,}$$

$P(x,y): x < y$

$Q(x,y): x > y$

To **disprove** S, should you prove:

✓

$$\forall x \in X, \forall y \in Y, P(x,y) \Rightarrow \neg Q(x,y)$$

What about

✗

$$\forall x \in X, \forall y \in Y, \neg(P(x,y) \Rightarrow Q(x,y))$$

$P(x,y) \wedge \neg Q(x,y)$

$X = Y = \mathbb{N}$ $\quad P(x,y): x < y$

$Q(x,y): x \mid y$

Explain why, or why not.

Define $T(n)$ by:

$$\forall n \in \mathbb{N} \qquad T(n) \Leftrightarrow \exists i \in \mathbb{N}, n = 7i + 1.$$

Take some scrap paper, **don't** write your name on it, and fill in as much of the proof of the following claim as possible:

$$\forall n \in \mathbb{N}, T(n) \Rightarrow T(n^2)$$

Now fill in as much of the **disproof** of the following claim as possible:

$$\forall n \in \mathbb{N}, T(n^2) \Rightarrow T(n)$$

Notes