CSC165 winter 2013

Mathematical expression

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Course notes, chapter 3

Outline

non-boolean functions

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non-boolean functions

Take care when expressing a proof about a function that returns a non-boolean value, such as a number:

$\lfloor x \rfloor$ *is the largest integer* $\leq x$.

Now prove the following statement (notice that we quantify over $x \in \mathbb{R}$, not $\lfloor x \rfloor \in \mathbb{R}$:

$$\forall x \in \mathbb{R}, \lfloor x \rfloor < x + 1$$

using more of the definition

You may have been disappointed that the last proof used only part of the definition of floor. Here's a symbolic re-writing of the definition of floor:

$$\forall x \in \mathbb{R} \qquad y = \lfloor x \rfloor \Leftrightarrow y \in \mathbb{Z} \wedge y \leq x \wedge (\forall z \in \mathbb{Z}, z \leq x \Rightarrow z \leq y)$$

The full version of the definition should prove useful to prove:

$$\forall x \in \mathbb{R}, \lfloor x \rfloor > x - 1$$

proving something false

Define a sequence:

$$\forall n \in \mathbb{N} \qquad a_n = \lfloor n/2 \rfloor$$

(of course, if you treat "/" as integer division, there's no need to take the floor. Now consider the claim:

$$\exists i \in \mathbb{N}, \forall j \in \mathbb{N}, j > i \Rightarrow a_j = a_i$$

The claim is false. Disprove it.

proof by cases

Sometimes your argument has to split to take into account possible properties of your generic element:

$$\forall n \in \mathbb{N}, n^2 + n \text{ is even}$$

A natural approach is to factor $n^2 + n$ as $n(n + 1)$, and then consider the case where n is odd, then the case where n is even.

proof about limits

In proving this claim you can't control the value of ε or y, but you can craft δ to make things work out.

$$\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+, \forall y \in \mathbb{R}, |y - \pi| < \delta \Rightarrow |y^2 - \pi^2| < \varepsilon$$

The claim is true. The proof format should be already familiar to you. A good approach is to fill in as much as possible, leaving the actual value of δ out until you have more intuition about it.

use uniqueness

Suppose you have a predicate of the natural numbers:

$$\forall n \in \mathbb{N} \qquad S(n) \Leftrightarrow \exists k \in \mathbb{N}, n = 7k + 3$$

Is $S(3 \times 3)$ true? How do you prove that? It's useful to check out the remainder theorem from the sheet of mathematical prerequisites.

get wrong right

Be careful proving a claim false. Consider the claim, for some suitably defined X, Y and P, Q:

$$S: \qquad \forall x \in X, \forall y \in Y, P(x,y) \Rightarrow Q(x,y)$$

To **disprove** S, should you prove:

$$\forall x \in X, \forall y \in Y, P(x,y) \Rightarrow \neg Q(x,y)$$

What about

$$\forall x \in X, \forall y \in Y, \neg(P(x,y) \Rightarrow Q(x,y))$$

Explain why, or why not.

Define $T(n)$ by:

$$\forall n \in \mathbb{N} \qquad T(n) \Leftrightarrow \exists i \in \mathbb{N}, n = 7i + 1.$$

Take some scrap paper, **don't** write your name on it, and fill in as much of the proof of the following claim as possible:

$$\forall n \in \mathbb{N}, T(n) \Rightarrow T(n^2)$$

Now fill in as much of the **disproof** of the following claim as possible:

$$\forall n \in \mathbb{N}, T(n^2) \Rightarrow T(n)$$

Notes