

CSC165 winter 2013

Mathematical expression

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Course notes, chapter 2–3



Outline

implication as disjunction

mixed quantifiers

proof

notes



implication two ways

The result of the following truth table is useful enough to bear restating:

P	Q	$P \Rightarrow Q$	$\neg P \vee Q$
T	T		
T	F		
F	T		
F	F		



bi-implication

Translate bi-implication into the conjunction of two disjunctions:

$$(P \Rightarrow Q) \wedge (Q \Rightarrow P)$$

Now change your expression for bi-implication into the disjunction of two conjunctions (use the some of the equivalences from a few slides ago)

What's the negation of bi-implication? How would you explain it in English?

transitivity

What does the following statement mean, when you interpret it as a venn diagram?

$$\forall x \in X, (P(x) \Rightarrow Q(x)) \wedge (Q(x) \Rightarrow R(x))$$

For another insight, negate the following statement, and simplify it by transforming implications into disjunctions:

$$((P \Rightarrow Q) \wedge (Q \Rightarrow R)) \Rightarrow (P \Rightarrow R)$$

for all, one...one for all

What's the difference between these two claims:

$$\forall x \in L1, \exists y \in L2, x + y = 5$$

$$\exists y \in L2, \forall x \in L1, x + y = 5$$

```
def P(x,y) : return x + y == 5
```

```
L1 = L2 = [1, 2, 3, 4]
```

```
def forallExists(P, L1, L2) :
```

```
    return False not in [True in [P(x,y) for y in L2] for x in L1]
```

```
def existsForall(P, L1, L2) :
```

```
    return True in [False not in [P(x,y) for x in L2] for y in L1]
```



Can you switch $\forall \varepsilon \in \mathbb{R}^+$ with $\exists \delta \in \mathbb{R}^+$ without altering the truthfulness of the statement below?

$$\forall x \in \mathbb{R}, \forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+, |x - 0.6| < \delta \Rightarrow |x^2 - 0.36| < \varepsilon$$

(you can!). How about:

$$\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+, \forall x \in \mathbb{R}, |x - 0.6| < \delta \Rightarrow |x^2 - 0.36| < \varepsilon$$

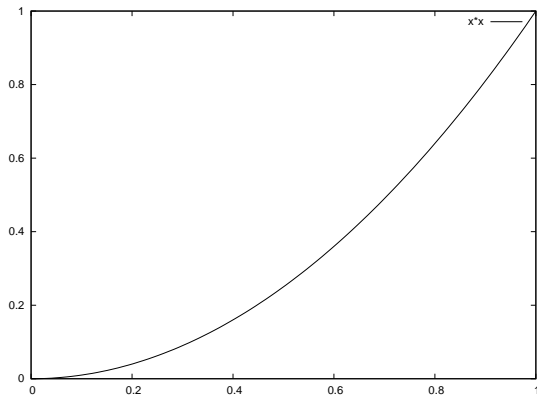
This latter is often written in a different form:

$$\lim_{x \rightarrow 0.6} x^2 = 0.36$$

First specify how close to 0.36 x^2 has to be (ε), then I can choose how close to 0.6 x must be (δ). If I choose δ first, can it work for all ε ?



graphically...



are we close to infinity yet?

What is meant by phrases such as “as x approaches (gets close to) infinity, x^2 increases without bound (sometimes ‘becomes infinite’)”?
Or even

$$\lim_{x \rightarrow \infty} x^2 = \infty$$

Look at the graph of x^2 . Do either x or x^2 ever reach infinity?

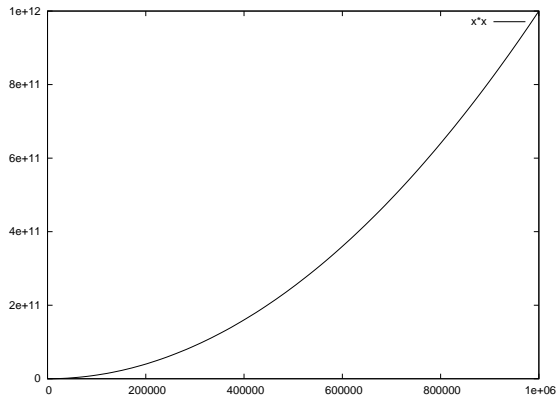
How about:

$$\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+, \forall x \in \mathbb{R}, x > \delta \Rightarrow x^2 > \varepsilon$$

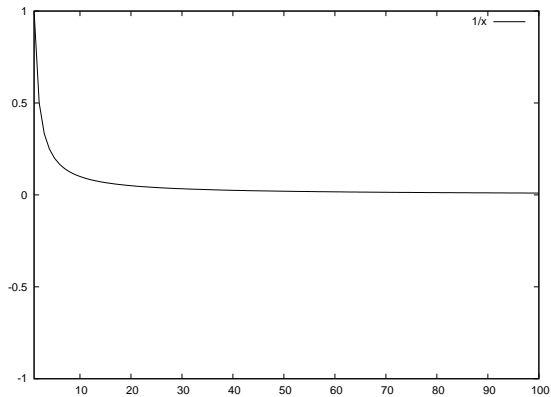
Getting “close” to infinity means getting far from (and greater than) zero. Once you have a specification for how far from zero x^2 must be (ε), you can come up with how far from zero x must be (δ). Can you choose a δ in advance that works for all ε ?



graph “approaching infinity”



asymptotic



double quantifiers

There are (at least) three ways to claim that a certain subset of the cartesian product $\mathbb{N} \times \mathbb{N}$, aka \mathbb{N}^2 is non-empty:

$$\exists m \in \mathbb{N}, \exists n \in \mathbb{N}, m^2 = n$$

$$\exists (m, n) \in \mathbb{N}^2, m^2 = n$$

$$\exists n \in \mathbb{N}, \exists m \in \mathbb{N}, m^2 = n$$

Whether we think of this as a statement about a subset of the cartesian product being empty, or a relation between non-empty subsets of \mathbb{N} , it is symmetrical.

There are (at least) three ways to claim that the entire cartesian product $\mathbb{N} \times \mathbb{N}$ has some property:

$$\forall m \in \mathbb{N}, \forall n \in \mathbb{N}, mn \in \mathbb{N}$$

$$\forall (m, n) \in \mathbb{N}^2, mn \in \mathbb{N}$$

$$\forall n \in \mathbb{N}, \forall m \in \mathbb{N}, mn \in \mathbb{N}$$

Again, the order in which we consider elements of an ordered pair doesn't change the logic.



a proof foretold

A proof communicates why and how you believe something to be true. You'll need to master two things:

1. Understand why you believe the thing is true. This step is messy, creative, but then increasingly precise to identify (and then strengthen) the weak parts of your belief.
2. Write up (express) why you believe the thing is true. Each step of your written proof should be justified enough to convince a skeptical peer. If you detect a gap in your reasoning, you may have to go back to step 1.

Although I present a great deal of symbolic notation, we will accept carefully-structured, precise English prose. The structure, however, is required, and is a main topic of Chapter 3.



find proof of universally-quantified \Rightarrow

$$\begin{array}{l} P \Rightarrow R \quad \wedge \\ R \Rightarrow Q \\ \Rightarrow P \Rightarrow Q \end{array}$$

To support a proof of a universally-quantified implication

$\forall x \in X, P(x) \Rightarrow Q(x)$, you usually need to use some already-proven statements and axioms (defined, or assumed, to be true for X). You hope to find a chain

$$\begin{array}{lll} C2.0 & \forall x \in X, P(x) & \Rightarrow R_1(x) \\ C2.1 & \forall x \in X, R_1(x) & \Rightarrow R_2(x) \\ & \vdots & \\ C2.n & \forall x \in X, \underline{R_n(x)} & \Rightarrow Q(x) \end{array} \quad \left. \begin{array}{l} \text{justification} \\ \text{required} \\ \text{(eg. math} \\ \text{pre reqs).} \end{array} \right\}$$

Such a chain shows in n steps that $P(x) \Rightarrow Q(x)$, by transitivity.

proof outline

$$\forall n \in \mathbb{N}, n \text{ odd} \Rightarrow n^2 \text{ odd}$$

More flexible format required in this course. Each link in the chain justified by mentioning supporting evidence in a comment beside it. Here are portions of an argument where scope of assumption is shown by indentation. A generic proof that $\forall x \in X, P(x) \Rightarrow Q(x)$ might look like:

Assume $x \in X$ # x is generic; what I prove applies to all of X

Assume $P(x)$. # Antecedent. Otherwise, $\neg P(x)$ means we get the implication for free.

Then $R_1(x)$ # by previous result

C2.0, $\forall x \in X, P(x) \Rightarrow R_1(x)$

Then $R_2(x)$ # by previous result

C2.1, $\forall x \in X, R_1(x) \Rightarrow R_2(x)$

\vdots

Then $Q(x)$ # by previous result

C2.n, $\forall x \in X, R_n(x) \Rightarrow Q(x)$

Then $P(x) \Rightarrow Q(x)$ # I assumed antecedent, got consequent
(aka introduced \Rightarrow)

Then $\forall x \in X, P(x) \Rightarrow Q(x)$ # reasoning works for all $x \in X$



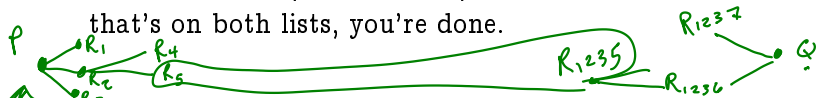
tracking the wiley chain of results



The hard part is finding that chain of implications. Here are two models for your search (they are equivalent).



- bubble search: As a venn diagram, you are search for a chain of supersets from P to Q . Work forwards (supersets of P) and backwards (subsets of Q). As soon as you find a set that's on both lists, you're done.



- tree search: As a directed graph, you are searching for a path from P to Q . Work forwards (consequents of P) and backwards (antecedents of Q). As soon as you find a predicate that's on both lists, you're done.



chains with \wedge or \vee

Chains of antecedents consequents break up in asymmetrical ways. Use truth tables, venn diagrams, or rules for manipulating predicates to show

$$((P \Rightarrow R_1) \wedge (P \Rightarrow R_2)) \Leftrightarrow (P \Rightarrow (R_1 \wedge R_2))$$

good exercises (Venn truth tables)

Notice that things switch when the conjunction is at the other end of the implication

$$((R_1 \Rightarrow Q) \wedge (R_2 \Rightarrow Q)) \Leftrightarrow ((R_1 \vee R_2) \Rightarrow Q)$$



an odd example

The square of an odd number is odd. Prove:

$$\forall n \in \mathbb{N}, n \text{ odd} \Rightarrow n^2 \text{ odd} .$$

$$\forall n \in \mathbb{N}, n \text{ odd} \Rightarrow n^2 \text{ odd}$$



a real inequality

Prove that for every pair of non-negative real numbers (x, y) , if x is greater than y , then the geometric mean, \sqrt{xy} is less than the arithmetic mean, $(x + y)/2$.



Notes

$$\forall n \in \mathbb{N}, n \text{ odd} \Rightarrow n^2 \text{ odd}$$

Proof:

Assume $n \in \mathbb{N}$ # or "typical/generic natural number"

Assume n odd # antecedent.

Then $\exists k \in \mathbb{N}, n = 2k+1$ # definition of
(Pick $k \in \mathbb{N}, n = 2k+1$) # odd element of \mathbb{N} .
optional -

$$\text{Then } n^2 = 4k^2 + 4k + 1 \text{ # algebra.}$$

$$= 2(2k^2 + 2k) + 1 \text{ # more algebra}$$

$$\text{Then } n^2 = 2k' + 1, k' \in \mathbb{N}.$$

$k' = 2k^2 + 2k \in \mathbb{N}$, since $k \in \mathbb{N}, 2 \in \mathbb{N}$,
and \mathbb{N} is closed under $+$ and \times

Then n^2 is odd # by def, since $n^2 = 2k' + 1$
$k' \in \mathbb{N}$

Then $n \text{ odd} \Rightarrow n^2 \text{ odd}$ # introduced \Rightarrow

Then $\forall n \in \mathbb{N}, n \text{ odd} \Rightarrow n^2 \text{ odd}$ # introduction \forall

