CSC165 winter 2013

Mathematical expression

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Course notes, chapter 2

Outline

Quantifiers, continued

Sentences, symbols

Notes

more existential claims

How do you evaluate:

- Some employee earns over 80,000. false
- Some male employee earns less than 27,000. false.
- Some female employee earns over 42,000.

Employee	Gender	Salary
Al	male	60,000
Betty	female	500
Carlos	male	40,000
Doug	male	30,000
Ellen	female	50,000
Flo	female	20,000

evaluating quantified claims as sets/lists

Suppose E is a list of employees, M is a list of male employees, F is a list of female employees, and O is a list of employees earning over 42,000. Explain how to use quant1–quant4 to evaluate them:

- All employees earn over 42,000 — quant1(O, M)
- Some female employee earns over 42,000
- Some male employee does not earn over 42,000
- All male employees does not earn over 42,000

quant1(M, O)

quant2(F, O)

quant3(E, O)

quant4(M, O)

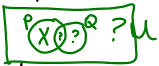
```
def quant1(L1, L2) :
    return False in [x in L2 for x in L1]
def quant2(L1, L2) :
    return True in [x in L2 for x in L1]
def quant3(L1, L2) :
    return False not in [x in L2 for x in L1]
def quant4(L1, L2) :
    return True not in [x in L2 for x in L1]
```

universal/existential duality

$P, Q,$ inside U

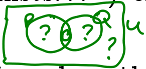
Here's an anti-symmetrical pattern to evaluating quantified claims:

- To verify a universal ("for all...") claim, show there is no counter-example. $\forall p \in P, p \in Q$



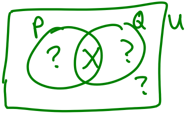
- To falsify a universal claim, find at least one counter-example. $\forall p \in P, p \in Q$

- To verify an existential ("exists...") claim, show there is at least one example.



$\exists p \in P, p \in Q$

- To falsify an existential claim, show there are no examples.



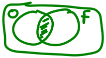
$\exists p \in P, p \in Q$

quantifiers as claims about sets

Think of quantification in terms of sets, so E is the set of employees, M is the set of male employees, F is the set of female employees, and O is the set of employees earning over 42,000. Express the following in terms of set operations (subsets, complements, etc.):

E $\quad$ O $\quad$ $E \subseteq O$

- All employees earn over 42,000 $\quad$ $F \not\subseteq \overline{O}$
- Some female employee earns over 42,000 $\quad$ $\exists e \in F, e \in O$



$F \cap O \neq \emptyset$

- Some male employee earns over 42,000 (M, O) $\quad$ $M \not\subseteq \overline{O}$
- All male employees earn over 42,000 (M, O) $\quad$ $\overline{O}$

$M \subseteq O$

55 got it, 18 got something else (about 3:1).

sentences

We'll use **sentence** to refer to expressions that are structured to evaluate to either true or false. Sometimes key objects in a sentence have not been specified, so the sentence is **open**, and we may not be able to evaluate it:

The employee earns over 55,000.
Every employee makes less than 55,000.

Quantifying an unspecified variable may change an open sentence (about some unspecified element) to a **statement** — an expression that can be evaluated to true or false.

symbols

Using symbols such as `M` to stand for the set of male employees, and `O` to stand for employees earning over 42,000 allows us to abstract away details and focus on the set relationship, whether $M \subseteq O$ or not.

We extend the symbolism in order to emphasize the connection between the set L (employees earning less than 55,000) and the boolean function that indicates whether something is in L:

$$L(x) : x \in L$$

Notice how similar this is to the definition of a boolean function (the keyword `def` would make it even more so). The argument x shows us how the argument is used in the definition. We can't $L(x)$ until we know what x is bound to — $L(\text{Al})$ evaluates differently from $L(\text{Carlos})$.

universally quantified sentence

Change open sentence $L(x)$ into a statement by universally quantifying it. This operation is used often enough that there is a symbol provided for convenience:

$\forall$ *employees, the employee makes less than 55,000.*
$\forall$ *employees* x, x *makes less than 55,000.*
$\forall x \in E, L(x).$

anti-symmetrically...

The corresponding existential statement about employees earning less than 55,000:

$$\exists x \in E, L(x)$$

...is not a statement about an element x, but about the set $E \cap L$.

Notes