# CSC165, Winter 2013 Assignment 1 sample solution 

1. Suppose $A$ is the set of aphorisms, $D(h)$ means $h$ is dodecahedral, and $C(h)$ means $h$ is catalytic (you needn't worry about the meaning of aphorism, dodecahedral, or catalytic for the rest of this question). Write the negation of each of the following sentences in English and in symbolic form.
(a) Every aphorism is catalytic unless it is dodecahedral.

Solution: Some aphorism is neither catalytic nor dodecahedral.

$$
\exists a \in A, \neg C(a) \wedge \neg D(a)
$$

(b) No aphorism is both catalytic and dodecahedral.

Solution: Some aphorism is both catalytic and dodecahedral.

$$
\exists a \in A, C(a) \wedge D(a)
$$

(c) All aphorisms that are not catalytic are dodecahedral.

Solution: Some aphorism that is not catalytic is also not dodecahedral.

$$
\exists a \in A, \neg C(a) \wedge \neg D(a)
$$

(d) Some dodecahedral aphorisms are catalytic.

Solution: No dodecahedral aphorism is catalytic.

$$
\forall a \in A, D(a) \Rightarrow \neg C(a)
$$

(e) Every aphorism is catalytic if, and only if, it is dodecahedral.

Solution: Some aphorism is either catalytic and not dodecahedral, or not catalytic and dodecahedral.

$$
\exists a \in A,(C(a) \wedge \neg D(a)) \vee(\neg C(a) \wedge D(a))
$$

2. Consider the sentence:
$S$ : For all triples of natural numbers $m, n, p$, if $p$ is prime and $p$ divides $m n$, then $p$ divides $m$ or $p$ divides $n$.

Each of the statements below is equivalent to either the converse, the contrapositive, or the negation of $S$. You must decide which label fits each statement, and explain your thinking.

Solution (preface): Recall that the contrapositive is equivalent to $S$ itself. Here's $S$ expressed symbolically, using $P(x)$ means $x$ is prime, and $a \mid b$ means $a$ divides $b$ :

$$
\forall m, n, p \in \mathbb{N},(P(p) \wedge p \mid m n) \Rightarrow(p|m \vee p| n)
$$

(a) For some triples of natural numbers $m, n, p$, neither $m$ nor $n$ is divisible by $p$, yet $p$ is prime and $p$ divides $m n$.

Solution: Written symbolically, this is:

$$
\exists m, n, p \in \mathbb{N}, \neg(p|m \vee p| n) \wedge P(p) \wedge p \mid m n
$$

This is the negation of $S$, since the negation of the universal yields the existential, and the negation of the implication yields the antecedent conjoined with the negation of the consequent.
(b) For every triple of natural numbers $m, n, p$, if $p$ divides neither $m$ nor $n$, then $p$ is not prime or $p$ doesn't divide $m n$.

Solution: Written symbolically this is:

$$
\forall m, n, p \in \mathbb{N}, \neg(p|m \vee p| n) \Rightarrow \neg(P(p) \wedge p \mid m n)
$$

The negated consequent implies the negated antecedent, so this is the contrapositive, and equivalent to $S$ (the quantification is unchanged also).
(c) For every triple of natural numbers $m, n, p$, if $p$ is not prime or $p$ doesn't divide $m n$, then $p$ divides neither $m$ nor $n$.
Solution: Symbolically, using De Morgan's Law, this is:

$$
\forall m, n, p \in \mathbb{N}, \neg(P(p) \wedge p \mid m n) \Rightarrow \neg(p|m \vee p| n)
$$

The negated antecedent implies the negated consequent, which is the converse of the negated consequent implying the negated antecedent, so this is the converse of the contrapositive of $S$. Since the contrapositive of $S$ is equivalent to $S$, its converse is equivalent to the converse of $S$.
(d) There is a triple of natural numbers $m, n, p$, such that $p$ is prime, $p$ divides $m n$, and $p$ divides neither $m$ nor $n$.
Solution: Symbolically, adding some parentheses for clarity:

$$
\exists m, n, p \in \mathbb{N},(P(p) \wedge p \mid m n) \wedge \neg(p|m \vee p| n)
$$

This is the antecedent conjoined with the negation of the consequent, and the quantifier is existential, so this is the negation of $S$.
3. Suppose $X$ is a set that contains developers, and projects. MegaMoth is the name of a project, and Codefinger is the name of one of the developers. Several predicates are defined on $X$ : $D(x)$ means $x$ is a developer, $P(x)$ means $x$ is a project, $M(x, y)$ means $x$ manages $y, W(x, y)$ means $x$ works on $y$, $E(x, y)$ means $x$ equals $y$, and $I(x, y)$ means $x$ is more important than $y$. Use the set $X$, the predicates and constants above, along with the logical connectives you have learned, to either translate an English sentence into symbolic form, or a symbolic sentence into English.
(a) There is exactly one developer in $X$ who is more important than Codefinger

Solution: Once you've done this a couple of times, you'll be allowed to use the short form $\exists$ ! for "there exists a unique".

$$
\exists x \in X, D(x) \wedge I(x, \text { Codefinger }) \wedge(\forall y \in X,(I(y, \text { Codefinger }) \wedge D(y)) \Rightarrow E(y, x))
$$

(b) $\forall x \in X, \forall y \in X, \forall z \in X, \forall w \in X,(W(x, y) \wedge W(z, w) \wedge I(y, w)) \Rightarrow \neg M(z, x)$

Solution: Nobody who works on a less important project manages somebody who works on a more important project.
(c) $\forall x \in X,(P(x) \wedge \exists y \in X,(W(y, x) \wedge(\forall w \in X, W(w, x) \Rightarrow E(w, y)))) \Rightarrow E(x$, MegaMoth $)$

Solution: The only project that has exactly one person working on it is MegaMoth.
(d) $\forall x \in X,(\forall y \in X, W(y$, MegaMoth $) \Rightarrow M(x, y)) \Rightarrow I(x$, Codefinger $)$

Solution: Anybody who manages everybody who works on Megamoth is more important than Codefinger.
(e) Codefinger has been a developer on every project in $X$ except MegaMoth.

Solution: I'll have to identify "has been a developer on" with "works on."

$$
\forall y \in X,(\neg E(y, \text { Megamoth }) \wedge P(x)) \Rightarrow W(\text { Codefinger }, y)
$$

4. Suppose $T$ is a set of natural numbers. Consider the statement:
$S 2$ : Every element of $T$ is an integer power of 2.
Which of the following statements imply $S 2$ ? Which of the following statements are implied by $S 2$ ? Explain.
(a) $T$ has at most 1 member that is odd.

Solution: $S 2$ implies this statement, since if $S 2$ is true, then every member of $T$ other than 1 is even. This statement doesn't imply $S 2$, since $\{3,6,12\}$ has at most one odd member, but several of them are not powers of 2 .
(b) If $i$ and $j$ are elements of $T$, and $j$ is greater than $i$, then $i$ divides $j$.

Solution: $S 2$ implies this statement, since if $S 2$ is true, then smaller elements of $T$ divide larger elements. However, this statement doesn't imply $S 2$, since (again) $\{3,6,12\}$ satisfies this statement but not $S 2$.
(c) The only prime number that divides elements of $T$ is 2 .

Solution: $S 2$ and this statement are equivalent. If every element of $T$ is a power of 2 , then the only prime factor of elements of $T$ is 2 , and conversely.
(d) $T$ has no elements.

Solution: This statement implies $S 2$ : every element of the empty set is a power of 2. However, $S 2$ does not imply this statement, since $\{2\}$ satisfies $S 2$ without being empty.
(e) $T=\{4,32,128\}$.

Solution: This statement implies $S 2$, since every element of $T$ is a power of 2 . However, $S 2$ doesn't imply this statement, since there are plenty of sets consisting of powers of 2 that are different from $\{4,32,128\}$.

