CSC165 fall 2014

Mathematical expression

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Course notes, chapter 2–3

Outline

mixed quantifiers

proof

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annotated slides

transitivity

What does the following statement mean, when you interpret it as a venn diagram?

$$\forall x \in X, (P(x) \Rightarrow Q(x)) \wedge (Q(x) \Rightarrow R(x))$$

For another insight, negate the following statement, and simplify it by transforming implications into disjunctions:

$$((P \Rightarrow Q) \wedge (Q \Rightarrow R)) \Rightarrow (P \Rightarrow R)$$

for all, one. . . one for all

What's the difference between these two claims:

$$\forall x \in S1, \exists y \in S2, x + y = 5$$
$$\exists y \in S2, \forall x \in S1, x + y = 5$$

```
S1 = S2 = {1, 2, 3, 4}

def forall_exists(S1, S2):
    return all({any({x+y == 5 for y in S2}) for x in S1})

def exists_forall(S1, S2):
    return any({all({x+y == 5 for y in S1}) for x in S2})

if __name__ == '__main__':
    print(forall_exists(S1, S2))
    print(exists_forall(S1, S2))
```

Can you switch $\forall e \in \mathbb{R}^+$ with $\exists d \in \mathbb{R}^+$ without altering the truthfulness of the statement below?

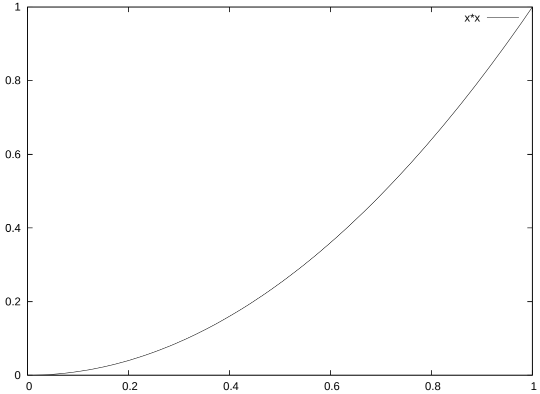
$$\forall e \in \mathbb{R}^+, \exists d \in \mathbb{R}^+, \forall x \in \mathbb{R}, |x - 0.6| < d \Rightarrow |x^2 - 0.36| < e$$

This latter is often written in a different form:

$$\lim_{x \to 0.6} x^2 = 0.36$$

First specify how close to 0.36 x^2 has to be (e), then I can choose how close to 0.6 x must be (d). If I choose d first, can it work for all e?

graphically...



are we close to infinity yet?

What is meant by phrases such as "as x approaches (gets close to) infinity, x^2 increases without bound (sometimes 'becomes infinite')"? Or even

$$\lim_{x \to \infty} x^2 = \infty$$

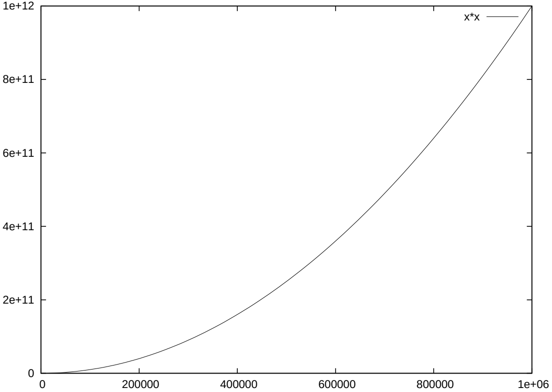
Look at the graph of x^2. Do either x or x^2 ever reach infinity?

How about:

$$\forall e \in \mathbb{R}^+, \exists d \in \mathbb{R}^+, \forall x \in \mathbb{R}, x > d \Rightarrow x^2 > e$$

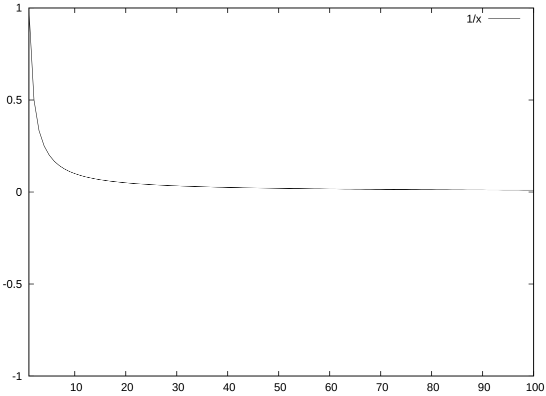
Getting "close" to infinity means getting far from (and greater than) zero. Once you have a specification for how far from zero x^2 must be (e), you can come up with how far from zero x must be (d). Can you choose a d in advance that works for all e?

graph "approaching infinity"



asymptotic

$$\forall e \in \mathbb{R}^+, \exists d \in \mathbb{R}^+, \forall x \in \mathbb{R}, x > d \Rightarrow |1/x| < e$$



double quantifiers

There are (at least) three ways to claim that a certain subset of the cartesian product $\mathbb{N} \times \mathbb{N}$, **aka** $\mathbb{N}^2$ is non-empty:

$$\exists m \in \mathbb{N}, \exists n \in \mathbb{N}, m^2 = n$$
$$\exists (m, n) \in \mathbb{N}^2, m^2 = n$$
$$\exists n \in \mathbb{N}, \exists m \in \mathbb{N}, m^2 = n$$

Whether we think of this as a statement about a subset of the cartesian product being empty, or a relation between non-empty subsets of $\mathbb{N}$, it is symmetrical.
There are (at least) three ways to claim that the entire cartesian product $\mathbb{N} \times \mathbb{N}$ has some property:

$$\forall m \in \mathbb{N}, \forall n \in \mathbb{N}, mn \in \mathbb{N}$$
$$\forall (m, n) \in \mathbb{N}^2, mn \in \mathbb{N}$$
$$\forall n \in \mathbb{N}, \forall m \in \mathbb{N}, mn \in \mathbb{N}$$

Again, the order in which we consider elements of an ordered pair doesn't change the logic.

a proof foretold

A proof communicates why and how you believe something to be true. You'll need to master two things:

1. Understand why you believe the thing is true. This step is messy, creative, but then increasingly precise to identify (and then strengthen) the weak parts of your belief.

2. Write up (express) why you believe the thing is true. Each step of your written proof should be justified enough to convince a skeptical peer. If you detect a gap in your reasoning, you may have to go back to step 1.

Although I present a great deal of symbolic notation, we will accept carefully-structured, precise English prose. The structure, however, is required, and is a main topic of Chapter 3.

find proof of universally-quantified $\Rightarrow$

To support a proof of a universally-quantified implication $\forall x \in X, P(x) \Rightarrow Q(x)$, you usually need to use some already-proven statements and axioms (defined, or assumed, to be true for X). You hope to find a chain

$$\begin{array}{lrcl}
C2.0 & \forall x \in X, P(x) & \Rightarrow & R_1(x) \\
C2.1 & \forall x \in X, R_1(x) & \Rightarrow & R_2(x) \\
 & & \vdots & \\
C2.n & \forall x \in X, R_n(x) & \Rightarrow & Q(x)
\end{array}$$

Such a chain shows in n steps that $P(x) \Rightarrow Q(x)$, by transitivity.

proof outline

More flexible format required in this course. Each link in the chain justified by mentioning supporting evidence in a comment beside it. Here are portions of an argument where scope of assumption is shown by identation. A generic proof that $\forall x \in X, P(x) \Rightarrow Q(x)$ might look like:

Assume $x \in X$ # x is generic; what I prove applies to all of X

Assume $P(x)$. # Antecedent. Otherwise, $\neg P(x)$ means we get the implication for free.

Then $R_1(x)$ # by previous result
$C2.0, \forall x \in X, P(x) \Rightarrow R_1(x)$
Then $R_2(x)$ # by previous result
$C2.1, \forall x \in X, R_1(x) \Rightarrow R_2(x)$

$\vdots$

Then $Q(x)$ # by previous result
$C2.n, \forall x \in X, R_n(x) \Rightarrow Q(x)$

Then $P(x) \Rightarrow Q(x)$ # I assumed antecedent, got consequent (aka introduced $\Rightarrow$)

Then $\forall x \in X, P(x) \Rightarrow Q(x)$ # reasoning works for all $x \in X$

tracking the wiley chain of results

The hard part is finding that chain of implications. Here are two models for your search (they are equivalent).

- **bubble search**: As a venn diagram, you are search for a chain of supersets from P to Q. Work forwards (supersets of P) and backwards (subsets of Q). As soon as you find a set that's on both lists, you're done.

- **tree search**: As a directed graph, you are searching for a path from P to Q. Work forwards (consequents of P) and backwards (antecedents of Q). As soon as you find a predicate that's on both lists, you're done.

chains with $\wedge$ or $\vee$

Chains of antecedents consequents break up in asymmetrical ways. Use truth tables, venn diagrams, or rules for manipulating predicates to show

$$((P \Rightarrow R_1) \wedge (P \Rightarrow R_2)) \Leftrightarrow (P \Rightarrow (R1 \wedge R_2))$$

Notice that things switch when the conjunction is at the other end of the implication

$$((R_1 \Rightarrow Q) \wedge (R_2 \Rightarrow Q)) \Leftrightarrow ((R_1 \vee R_2) \Rightarrow Q)$$

an odd example

The square of an odd number is odd. Prove:

$$\forall n \in \mathbb{N}, n \text{ odd } \Rightarrow n^2 \text{ odd }.$$

a real inequality

Prove that for every pair of non-negative real numbers (x, y), if x is greather than y, then the geometric mean, $\sqrt{xy}$ is less than the arithmetic mean, $(x + y)/2$.

Notes

annotated slides

- monday's annotated slides
- wednesday's annotated slides
- friday's annotated slides