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Mathematical expression

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Course notes, chapter 2

Outline

quantifiers, continued

sentences, symbols

implication

logical connectives

conjunction, disjunction

negation

notes

annotated slides

more existential claims

How do you evaluate:

- Some employee earns over 80,000.
- Some male employee earns less than 27,000.
- Some female employee earns over 42,000.

Employee	Gender	Salary
Al	male	60,000
Betty	female	500
Carlos	male	40,000
Doug	male	30,000
Ellen	female	50,000
Flo	female	20,000

evaluating quantified claims as sets

Suppose `E` is a set of employees, `M` is a set of male employees, `F` is a set of female employees, and `O` is a set of employees earning over 42,000. Explain how to use q0–q3 to evaluate them:

- All employees earn over 42,000
- Some female employee earns over 42,000
- Some male employee does not earn over 42,000
- All male employees does not earn over 42,000

```
def q0(S1, S2):
    return not all({x in S2 for x in S1})
def q1(S1, S2) :
    return any({x in S2 for x in S1})
def q2(S1, S2) :
    return all({x in S2 for x in S1})
def q3(S1, S2) :
    return not any({x in S2 for x in S1})
```

universal/existential duality

Here's an anti-symmetrical pattern to evaluating quantified claims:

- To **verify** a universal ("for all...") claim, show there is no **counter-example**.
- To **falsify** a universal claim, find at least one **counter-example**.
- To **verify** an existential ("exists...") claim, show there is at least one **example**.
- To **falsify** an existential claim, show there are no **examples**.

quantifiers as claims about sets

Think of quantification in terms of sets, so `E` is the set of employees, `M` is the set of male employees, `F` is the set of female employees, and `O` is the set of employees earning over 42,000. Express the following in terms of set operations (subsets, intersections, unions, complements, etc.):

- All employees earn over 42,000
- Some female employee earns over 42,000
- Some male employee earns over 42,000
- All male employees earn over 42,000

sentences

We'll use **sentence** to refer to expressions that are structured to evaluate to either true or false. Sometimes key objects in a sentence have not been specified, so the sentence is **open**, and we may not be able to evaluate it:

The employee earns over 55,000.
Every employee makes less than 55,000.

Quantifying an unspecified variable may change an open sentence (about some unspecified element) to a **statement** — an expression that can be evaluated to true or false.

symbols

Using symbols such as `M` to stand for the set of male employees, and `O` to stand for employees earning over 42,000 allows us to abstract away details and focus on the set relationship, whether $M \subseteq O$ or not.

We extend the symbolism in order to emphasize the connection between the set L (employees earning less than 55,000) and the boolean function that indicates whether something is in L:

$$L(x) : x \in L$$

Notice how similar this is to the definition of a boolean function (the keyword `def` would make it even more so). The argument x shows us how the argument is used in the definition. We can't $L(x)$ until we know what x is bound to — $L(\mathrm{Al})$ evaluates differently from $L(\mathrm{Carlos})$.

universally quantified sentence

Change open sentence $L(x)$ into a statement by universally quantifying it. This operation is used often enough that there is a symbol provided for convenience:

$\forall$ *employees, the employee makes less than 55,000.*
$\forall$ *employees* x, x *makes less than 55,000.*
$\forall x \in E, L(x)$.

anti-symmetrically...

The corresponding existential statement about employees earning less than 55,000:

$$\exists x \in E, L(x)$$

...is not a statement about an element x, but about the set $E \cap L$ not being empty, or E not being a subset of $\overline{L}$.

implication

There's a couple of ways to expression the **implication**

> **if** *an employee is male*, **then** *he earns less than 55,000.*

This could accurately be expressed using universal quantification by restricting the set we are considering:

$$\forall x \in E \cap M, L(x)$$

It's sometimes convenient to separate the "male implies less than 55,000" from the domain "employee" — perhaps seeing how the rule holds up in the larger set H of humans, or the smaller set S of short employees. The form "if P, then Q" is called **implication**.

verifying implication

Which of the following are a counter-example to "if the employee is male, then he earns less than 55,000"?

- Carlos?
- Ellen?
- Al?
- Gwen?

Employee	Gender	Salary
Al	male	60,000
Betty	female	500
Carlos	male	40,000
Doug	male	30,000
Ellen	female	50,000
Flo	female	20,000
Gwen	female	95,000

nomenclature

In implication "If P, then Q" we call P the **antecedent** and Q the **consequent**. Sometimes, in natural language an implication goes both ways:

If you eat your vegetables, then you can have dessert.

...but in logic, we allow the case where you don't eat your vegetables and still eat dessert to be consistent with the implication (what is the lone counter-example to this implication?)

Even true implication doesn't give you **causality**:

If it rains today, the sun will rise tomorrow.

implication information

Here's a universally-quantified implication, where E is the set of employees, F the set of female employees, and L the set of employees earning less than 55,000:

$$\forall x \in E, \textit{ if } F(x), \textit{ then } L(x).$$

If the implication is true, what can you deduce about the following sets:

1. F, the set of female employees?
2. L, the set of employees earning less than 55,000?
3. $\overline{F}$, the set of non-female employees?
4. $\overline{L}$, the set of employees earning at least 55,000?

If you could add a new employee, what gender and salary combination would you pick in order to falsify the implication?

a glyph of its own...

Implication is used frequently enough to deserve its own symbol. The universally-quantified implication from the previous slide could be written:

$$\forall x \in E, F(x) \Rightarrow L(x)$$

Reverse the direction, and you have the **converse** of the original implication.

$$\forall x \in E, L(x) \Rightarrow F(x)$$

What connection is there between the truth of an implication and the truth of its converse? Explain.

negation and contrapositive

Another symbol, $\neg$, toggles the truth value of a statement. When we toggle **and** reverse an implication, we get its **contrapositive**. Compare the meanings of:

$$\forall x \in E, F(x) \Rightarrow L(x)$$
$$\forall x \in E, \neg L(x) \Rightarrow \neg F(x)$$

What information does each form give you in each of the four following cases:

1. When $x \in F$?
2. When $x \notin F$?
3. When $x \in L$?
4. When $x \notin L$?

What would a counter-example to each form be?

numerical example

Define $P(n)$: n is a multiple of 4, and $Q(n)$: n^2 is a multiple of 4, and consider

$$\forall n \in \mathbb{N}, P(n) \Rightarrow Q(n)$$

What do the implication, converse, and contrapositive each tell you when

- n is a multiple of 4
- n is not a multiple of 4
- n^2 is a multiple of 4
- n^2 is not a multiple of 4

Which do you believe, and why?

“natural” language

Here are some ways of expressing implication, $P \Rightarrow Q$, in English. What's P and what's Q, in each case?

If nominated, I will not stand.

If you think I'm lying, then you're a liar!

Whenever I hear that song, I think about icecream.

Differentiability is sufficient for continuity.

Matching fingerprints and a motive are enough for guilt.

You can't stay enrolled in CSC165 without a pulse.

Successful programming requires skill.

I'll go only if you insist.

Don't knock it unless you've tried it.

vacuous truth

We've already separated implication from quantification, so we can make sense of

$$P(x) \Rightarrow Q(x)$$

It's true, except when $P(x)$ is true and $Q(x)$ is false. In particular, an implication is always true when the antecedent is false. For example, if your eyes wander to the consequent in

$$\forall x \in \mathbb{R}, x^2 - 2x + 2 = 0 \Rightarrow x > x + 5$$

...you could jump to the conclusion that the implication is false.

Vacuous truth works because there are no counterexamples. Another way of thinking about this is that the empty set is a subset of every other set.

All employees earning over 80 trillion dollars are female.
All employees earning over 80 trillion dollars are male.
All employees earning over 80 trillion dollars have mauve eyeballs and breathe ammonia.

equivalence

Suppose Al quits. Now consider the statement:

> *Every male employees earns between 25,000 and 45,000.*

Is the statement true? What about its converse?

Employee	**Gender**	**Salary**
Betty	female	500
Carlos	male	40,000
Doug	male	30,000
Ellen	female	50,000
Flo	female	20,000
Gwen	female	95,000

An employee is male if, and only if, that employee earns 25,000–45,000. This is a double implication, $P \Rightarrow Q$ and $Q \Rightarrow P$, or $P \Leftrightarrow Q$. Thought of as sets, they are equal (mutual subsets).

weird equivalence

How do you feel about

$$\forall x \in \mathbb{R}, x^2 - 2x + 2 = 0 \Leftrightarrow x > x + 5.$$

Break it into two implications:

$$\forall x \in \mathbb{R}, x^2 - 2x + 2 = 0 \Rightarrow x > x + 5.$$
$$\forall x \in \mathbb{R}, x > x + 5 \Rightarrow x^2 - 2x + 2 = 0.$$

The truth values are the same. English phrases:
P is necessary and sufficient for Q.
P is true exactly when Q is true.
P implies Q, and conversely.

idiom

Some expressions for restricting domains are more common than others.

- "Every D that is a P is also a Q." Usually $\forall x \in D, P(x) \Rightarrow Q(x)$. Less common $\forall x \in D \cap P, Q(x)$. What about $\forall x \in D, P(x) \wedge Q(x)$ ($\wedge$ means "and")?
- "Some D that is a P is also a Q." Usually $\exists x \in D, P(x) \wedge Q(x)$. Less common $\exists x \in D \cap P \cap Q$. What about $\exists x \in D, P(x) \Rightarrow Q(x)$?

conjunction: $\wedge$

Combine two statements by claiming they are both true with logical "and":

$A(x)$ and $B(x)$ (python keyword **and** works like this)

$A(x) \wedge B(x)$ ($\wedge$ is a symbol for "and")

As sets: $x \in A \cap B$

Notice that a conjunction is **false** if either part is false. "The employee makes less than 100,000 and more than 60,000," is true for Gwen, but false for Ellen.

Employee	Gender	Salary
Betty	female	500
Carlos	male	40,000
Doug	male	30,000
Ellen	female	50,000
Flo	female	20,000
Gwen	female	95,000

watch out for English "and"

Sometimes the English word "and" is used to smear some meaning over several components:

There is a pen and a telephone.

In the universe of objects, O, with predicates $P(x)$ (x is a pen) and $T(x)$ (x is a telephone), you could try to translate this as $\exists x \in O, P(x) \wedge T(x)$. What's a better translation into symbols?

Occasionally English usage of **and** will differ from logical usage even in mathematical material:

The solutions are $x < 10$ *and* $x > 20$
The solutions are $x < 20$ *and* $x > 10$

The first statements probably meant the union of the two sets, or the logical **or**. The second meant the intersection, so the logical **and** is appropriate.

disjunction: $\vee$

Combine two statements by claiming that at least one of them is true using **or** ($\vee$ in symbols).

$A(x)$ or $B(x)$ (the python keyword or works like this)

$A(x) \vee B(x)$ (in symbols)

$x \in A \cup B$ (as sets)

Notice the close connection between the symbols for conjunction and intersection, $\wedge$, $\cap$, and the symbols for disjunction and union, $\vee$, $\cup$. Coincidence? In any case, you may use it as a mnemonic.

"The employee is female or earns more than 35,000."

Employee	**Gender**	**Salary**
Betty	female	500
Carlos	male	40,000
Doug	male	30,000
Ellen	female	50,000
Flo	female	20,000

silly English tricks

In logic we use **or** generously, or inclusively, to mean something like "and/or". Sometimes we convey the **inclusive or** by saying something like "A or B, or both." Be aware that natural English sometimes uses or to mean "A or B, but not both" — something we'd call **exclusive or** in logic:

> *Either we play the game my way, or I'm taking my ball and going home.*

negation: $\neg$

Negate the statement "All employees earning over 110,000 are female." Usually prepending the word "Not" will work, and in logic we use the corresponding symbol $\neg$:

$$\neg(\forall e \in E, O(e) \Rightarrow F(e))$$

A good exercise is to "work" the negation $\neg$ as far into the statement as possible. The statement is true exactly when its negation is false.

The original statement is universally quantified, so it says something about an absence of counterexamples. The negation of the original statement should claim something about the presence of counterexamples.

special negation idiom

Negating implications is a common task. There are several equivalent ways of doing this, but some are more common than others. Try negating the following in such a way that the $\neg$ symbol applies to the "smallest possible" part of the expression:

$$\forall x \in X, P(x) \Rightarrow Q(x)$$

Now for symmetry, negate the following in such a way that the $\neg$ symbol applies to the "smallest possible" part of the expression:

$$\exists x \in X, P(x) \wedge \neg Q(x)$$

standard negation

Negated expressions have some standard transformations:

- $\neg\forall x \in X, \ldots \Leftrightarrow \exists x \in X, \neg \ldots$
- $\neg\exists x \in X, \ldots \Leftrightarrow \forall x \in X, \neg \ldots$
- $\neg(P(x) \Rightarrow Q(x)) \Leftrightarrow P(x) \wedge \neg Q(x)$
- $\neg(P(x) \wedge Q(x)) \Leftrightarrow P(x) \Rightarrow \neg Q(x)$ (has this become asymmetrical?)

Push the $\neg$ symbol "as far in" to the following expression as possible:

$$\neg(\forall x \in X, \exists y \in Y, P(x) \Rightarrow Q(y))$$

Notes

monday's annotated slides

wednesday's annotated slides

friday's annotated slides