CSC236 fall 2014
recursion, induction, correctness

Danny Heap
heap@cs.toronto.edu
BA4270 (behind elevators)
http://www.cdf.toronto.edu/~csc236h/fall/
416-978-5899

Using Introduction to the Theory of Computation,
Chapter 2
Outline

- gaussian multiplication
- binary search
- annotations
intuition about master theorem

\[ T(n) = \begin{cases} 
  c & n = 1 \\
  a_2 T(\lceil n/b \rceil) + a_1 T(\lfloor n/b \rfloor) + f(n) & n > 1
\end{cases} \]
Gauss’s trick

\[ xy = 2^n x_1 y_1 + x_0 y_0 + 2^{n/2} \left( (x_1 + x_0)(y_1 + y_0) - x_1 y_1 - x_0 y_0 \right) \]
Gauss’s payoff
lose one multiplication

1. divide each factor (roughly) in half
2. sum the halves
3. multiply the sum and the halves Gauss-wise
4. combine the products with shifts and adds
recursive binary search

def recBinSearch(x, A, b, e):
    if b == e:
        if x <= A[b]:
            return b
        else:
            return e + 1
    else:
        m = (b + e) // 2  # midpoint
        if x <= A[m]:
            return recBinSearch(x, A, b, m)
        else:
            return recBinSearch(x, A, m+1, e)
conditions, pre- and post-

- $x$ and elements of $A$ are comparable
- $e$ and $b$ are valid indices, $b \leq e$
- $A[b..e]$ is sorted non-decreasing

RecBinSearch($x, A, b, e$) terminates and returns index $p$

- $b \leq p \leq e + 1$
- $b < p \Rightarrow A[p - 1] < x$
- $p \leq e \Rightarrow x \leq A[p]$

(except for boundaries, returns $p$ so that $A[p - 1] < x \leq A[p]$)
precondition ⇒ termination and postcondition

Proof: induction on \( n = e - b + 1 \)

Base case, \( n = 1 \): Terminates because there are no loops or further calls, returns \( x \leq A[b = p] \iff p = b = e \) is returned. \( x > A[b = p - 1] \iff p = b + 1 \) returned, so postcondition satisfied. Notice that the choice forces if-and-only-if.

Induction step: Assume \( n > 1 \) and that the postcondition is satisfied for inputs of size \( 1 \leq k < n \) that satisfy the precondition. Call \( \text{RecBinSearch}(A,x,b,e) \) when \( n = e - b + 1 > 1 \). Since \( b < e \) in this case, the test on line 1 fails, and line 7 executes. Exercise: \( b \leq m < e \) in this case. There are two cases, according to whether \( x \leq A[m] \) or \( x > A[m] \).
Case 1: \( x \leq A[m] \)

- Show that IH applies to RBS\((x, A, b, m)\)
- Translate the postcondition to RBS\((x, A, b, m)\)

- Show that RBS\((x, A, b, e)\) satisfies postcondition
Case 2: $x > A[m]$

- Show that IH applies to $\text{RBS}(x, A, m+1, e)$
- Translate postcondition to $\text{RBS}(x, A, m+1, e)$

- Show that $\text{RBS}(x, A, b, e)$
what could go wrong?

- $m = \left\lfloor \frac{e+b}{2.0} \right\rfloor$

- $x < A[m]$

- ...

- Either prove correct, or find a counter-example
recursive and iterative
mergesort

MergeSort(A,b,e):
1. if b == e: return
2. m = (b + e) / 2  # integer division
3. MergeSort(A,b,m)
4. MergeSort(A,m+1,e)
   # merge sorted A[b..m] and A[m+1..e] back into A[b..e]
5. for i = b,...,e: B[i] = A[i]
6. c = b
7. d = m+1
8. for i = b,...,e:
   9. if d > e or (c <= m and B[c] < B[d]):
      10. A[i] = B[c]
      11. c = c + 1
   else:  # d <= e and (c > m or B[c] >= B[d])
      13. d = d + 1
conditions, pre- and post-

- $b$ and $e$ are natural numbers, $0 \leq b \leq e < \text{len}(A)$.
- elements of $A$ are comparable

- $A'[b..e]$ contains the same elements as $A[b..e]$, but sorted in non-decreasing order (use notation $A'$ for $A$ after calling MergeSort($A, b, e$)). All other elements of $A'$ are unchanged.
Proof of correctness of MergeSort(A, b, e)
by induction on $n = e - b + 1$ for all arrays of size $n$,
(precondition+execution) $\Rightarrow$ (termination+postcondition)

Base case, $1 = e - b + 1$: Assume MergeSort(A, b, e) is called
with $\text{len}(A) = 1$ preconditions satisfied. Then $0 \leq e \leq b \leq 0$,
so $e = b$, and the algorithm terminates with a (trivially)
sorted $A'$, satisfying the precondition.

Induction step: Assume $n \in \mathbb{N}$, $n > 1$, and for all natural
numbers $k$, $1 \leq k < n$, that MergeSort on all arrays of size $k$
that satisfy the precondition and run will terminate and satisfy
the postcondition. Assume MergSort(A, b, e) is executed and
$n = e - b + 1$. 
The test on line 1 fails, and $m$ is set to $(b + e)/2$, strictly less than $e$ (exercise).

Does the IH apply to $\text{MergeSort}(A,b,m)$ and $\text{MergeSort}(A,m+1,e)$? Translate the IH into postconditions for $\text{MergeSort}(A,b,m)$ and $\text{MergeSort}(A,m+1,e)$.

Now we need iterative correctness for the merge...
iterative correctness

partial correctness plus termination

- Preconditions plus termination imply the postcondition. Probably needs a loop invariant

- termination — construct a decreasing sequence in $\mathbb{N}$. 
annotated slides