CSC236 fall 2014

time complexity

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Using *Introduction to the Theory of Computation*,
Chapter 3
Outline
Number of binary strings without adjacent 0s

This is easy when $n = 0$ or $n = 1$. For $n > 1$ we have the possibility that the last bit added creates a forbidden 00.

The formula turns out to be related to $F(n)$, and it has the same annoying property $F(n)$ using the definition requires about $n$ calculations.
Closed form for $F(n)$?

No rabbit, no hat

The course notes present a proof by induction that

$$F(n) = \frac{\phi^n - (\hat{\phi})^n}{\sqrt{5}}, \quad \phi = \frac{1 + \sqrt{5}}{2}, \hat{\phi} = \frac{1 - \sqrt{5}}{2}$$

You can, and should, be able to work through the proof. The question remains, why did somebody think of $\phi$ and $\hat{\phi}$?
Closed form
...without rabbits

Start with the idea that $F(n)$ seems to increase by something close to a fixed ratio. Let's try calling that $r$, and $r$ has to satisfy:

$$r^n = r^{n-1} + r^{n-2} \implies r^2 = r + 1$$

There are two solutions to the quadratic equation: $\phi$ and $\hat{\phi}$, but what about the $1/\sqrt{5}$ factor?

If $\phi$ and $\hat{\phi}$ are solutions, so are linear combinations:

$$\alpha\phi^n + \beta\hat{\phi}^n = \alpha\phi^{n-1} + \beta\hat{\phi}^{n-1} + \alpha\phi^{n-2} + \beta\hat{\phi}^{n-2}$$
Rabbits, hats

Match up $\alpha$ and $\beta$ to solutions:

\[
\alpha \phi^0 + \beta \hat{\phi}^0 = 0 \quad \Rightarrow \quad \alpha = -\beta
\]

\[
\alpha \phi^1 + \beta \hat{\phi}^1 = 1 \quad \Rightarrow \quad \alpha (\phi - \hat{\phi}) = 1
\]
binary search

Recursive $T(n)$

def recBinSearch(x, A, b, e):
    if b == e:
        if x <= A[b]:
            return b
        else:
            return e + 1
    else:
        m = (b + e) // 2  # midpoint
        if x <= A[m]:
            return recBinSearch(x, A, b, m)
        else:
            return recBinSearch(x, A, m+1, e)
Lower bound on $T(n)$... by unwinding
Lower bound on $T(n)$... by proof
Upper bound on $T(n)$

trouble!
annotated slides