CSC236 fall 2014
structural induction

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Using Introduction to the Theory of Computation,
Chapter 4, Section 1.1
Outline

equivalence of induction principles

structural induction

notes
main induction principles

**Induction (I):**

\[ P(0) \land (\forall n \in \mathbb{N}, P(n) \Rightarrow P(n + 1))] \Rightarrow \forall n, P(n) \]

**Complete Induction (CI):**

\[ \forall n \in \mathbb{N}, (\forall k \in \{k \in \mathbb{N} \mid 0 \leq k < n\}, P(k)) \Rightarrow P(n)] \Rightarrow \forall n \in \mathbb{N}, P(n) \]

**Principle of Well Ordering (PWO):** Every non-empty subset of \( \mathbb{N} \) has a smallest element.
The cycle is proved in the text, here is one link. Suppose you believe MI, and you have shown for some property $P$:

$$\forall n \in \mathbb{N}, (\forall i \in \{i \in \mathbb{N} | 0 \leq i < n\}, P(i)) \Rightarrow P(n) \quad (1)$$

Now define a slightly different predicate:

$P'(n) : \forall i \in \{i \in \mathbb{N} | 0 \leq i \leq n\}, P(i)$, in other words, $P(i)$ is true up to and including $n$. Using only Induction prove $\forall n, P'(n)$:

**Base case:** Since we showed (1), and there are no natural numbers smaller than 0, we have $P'(0)$.

**Induction step:** Assume $n$ is an arbitrary natural number and that $P'(n)$ is true. It follows from (1) that $P(n + 1)$ is true, and hence $P'(n + 1)$ is true.
Define sets inductively
...so as to use induction on them later

Define $\mathcal{E}$: The smallest set such that

- $x, y, z \in \mathcal{E}$
- $e_1, e_2 \in \mathcal{E} \Rightarrow (e_1 + e_2), (e_1 - e_2), (e_1 \times e_2)$, and $(e_1 \div e_2) \in \mathcal{E}$.

Form some expressions in $\mathcal{E}$. Count the number of variables (symbols from $\{x, y, z\}$) and the number of operators (symbols from $\{+, \times, \div, -\}$). Make a conjecture.
### Structural induction

\( P(e) : \text{vr}(e) = \text{op}(e) + 1 \)

To prove that a property is true for all \( e \in \mathcal{E} \), parallel the recursive set definition:

- **Base case:** Show that the property is true for the simplest members, \( \{x, y, z\} \)

- **Induction step:** Show “inheritance”: if \( P(e_1) \) and \( P(e_2) \), then all possible combinations \( (e_1 + e_2) \), \( (e_1 - e_2) \), \( (e_1 \times e_2) \), and \( (e_1 \div e_2) \) have the property.

Conclude that the property is true of all elements of \( \mathcal{E} \).
Structural induction

\[ P(e) : \text{vr}(e) = \text{op}(e) + 1 \]

Prove \( \forall e \in \mathcal{E}, P(e) \)

Proof (using structural induction):

**Base Case** \( \text{vr}(x) = \text{vr}(y) = \text{vr}(z) = 1 \), and they each have 0 operators, so the claim holds for the basis: \( \text{vr}(e) = 1 = \text{op}(e) + 1 \), where \( e \in \{ x, y, z \} \).

**Induction Step** Assume that \( e_1, e_2 \) are typical elements of \( \mathcal{E} \), and that \( P(e_1) \) and \( P(e_2) \) hold. Must show that \( P((e_1 \circ e_2)) \) is true, where \( \circ \in \{ +, - \} \) and \( \text{vr}(e_1) = \text{vr}(e_2) = \text{vr}(e_1 \circ e_2) \).

Then \( e_1 \circ e_2 \) has the same variables as \( e_1 \) and \( e_2 \).
Structural induction

\[ P(e) : \text{vr}(e) = \text{op}(e) + 1 \]

Prove \( \forall e \in \mathcal{E}, P(e) \)

So \( \text{vr}((e_1 \circ e_2)) = \text{vr}(e_1) + \text{vr}(e_2) \)

\[ = \overline{\text{op}(e_1)} + 1 + \overline{\text{op}(e_2)} + 1 \]

\# by IH, \( P(e_1) \) and \( P(e_2) \)

\[ = \overline{\text{op}((e_1 \circ e_2))} + 1 + 2 \]

\# \( e_1 \circ e_2 \) has one more operator

\# than sum of \( \text{op}(e_1) \) and \( \text{op}(e_2) \)

\[ = \overline{\text{op}((e_1 \circ e_2))} + 1 \]

So \( P((e_1 \circ e_2)) \) follows from \( P(e_1), P(e_2) \)

Conclude \( P(e) \forall e \in \mathcal{E}, \) by structural induction
More structural induction

Define the height of $x$, $y$, or $z$ as 0, and $h((e_1 \odot e_2))$ as $1 + \max(h(e_1), h(e_2))$, if $e_1, e_2 \in \mathcal{E}$ and $\odot \in \{+, \times, \div, -\}$. What’s the connection between the number of variables and the height?

\[
\begin{array}{c|c|c}
\hline
h & v \\
\hline
0 & 1 \\
1 & 2 \\
2 & 3, 4 \\
3 & 8, \ldots, 14 \\
\hline
\end{array}
\]

**Conjecture**

$\forall r(e) \leq 2^h(e)$
More structural induction

\( P(e) : \text{vr}(e) \leq 2^{h(e)} \)

Prove \( \forall e \in E, P(e) \), using Structural Induction

Basis If \( e \in \{ \alpha, \beta, \gamma \} \), then \( \text{vr}(e) = 1 \) and
\[ 1 \leq 2^0 = 2^{h(e)} \], so \( P(e) \) holds.

Induction Step Assume \( e_1, e_2 \) are typical elements of \( E \).
Also, assume \( \text{vr}(e_1) \leq 2^{h(e_1)} \) and \( \text{vr}(e_2) \leq 2^{h(e_2)} \), that is \( \text{vr}(e_1) \leq 2^{h(e_1)} \).

Show: Must show \( P(\langle e_1, 0 e_2 \rangle) \), i.e., \( \text{vr}(\langle e_1, 0 e_2 \rangle) \leq 2^{h(\langle e_1, 0 e_2 \rangle)} \), for \( 0 \in \{ +, - \} \).

Then \( \text{vr}(\langle e_1, 0 e_2 \rangle) = \text{vr}(e_1) + \text{vr}(e_2) \)
\[ \leq 2^{h(e_1)} + 2^{h(e_2)} \] # by IH
\[ \leq 2^{h(\langle e_1, 0 e_2 \rangle) - 1} + 2^{h(\langle e_1, 0 e_2 \rangle) - 1} \]
\[ \leq h(\langle e_1, 0 e_2 \rangle) \geq 1 + h(\alpha) \text{ and } 1 + h(\gamma) \]
\[ = 2^{h(\langle e_1, 0 e_2 \rangle)} \]
More structural induction

\[ P(e) : \text{vr}(e) \leq 2^{h(e)} \]

so \( P((e, 0 \cdot e_2)) \) follows from \( P(e_1) \) and \( P(e_2) \).

Conclude \( \forall e \in E, P(e) \), by structural induction.
Recursive definition

Fibonacci sequence

This sequence comes up in applied rabbit breeding, the height of AVL trees, and the complexity of Euclid’s algorithm for the GCD:

\[ F(n) = \begin{cases} 
  n & \text{if } n < 2 \\
  F(n-2) + F(n-1) & \text{if } n \geq 2
\end{cases} \]

What is the sum of \( n \) Fibonacci numbers?

<table>
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<th>( n )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>( F(n) )</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td>8</td>
<td>13</td>
</tr>
<tr>
<td>( \sum F(n) )</td>
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<td>1</td>
<td>2</td>
<td>4</td>
<td>7</td>
<td>12</td>
<td>20</td>
<td>33</td>
</tr>
</tbody>
</table>
Fibonacci numbers

What is $\sum_{i=0}^{n} F(i)$?

Prove $\forall n \in \mathbb{N}$, $P(n)$ using complete induction.

Let $n \in \mathbb{N}$, and assume $\forall k \in \{k \in \mathbb{N} : 0 \leq k < n^2\}$, $P(k)$, that is for every natural number $k < n$, $\sum_{i=0}^{k} F(i) = F(k+2) - 1$ (induction hypothesis). Must show $P(n)$.

Case 1. $n = 0$

Case 2. Assume $n > 0$

Then $\sum_{i=0}^{n} F(i) = \left[\sum_{i=0}^{n-1} F(i)\right] + F(n)$

$= F(n-1+2) - 1 + F(n)$ (by IH),

$= F(n+1) + F(n) - 1$

$= F(n+2) - 1$ (by def $F$)

Since $n+2 > 1$, $n \in \mathbb{N}$
Number of binary strings without adjacent 0s

This is easy when $n = 0$ or $n = 1$. For $n > 1$ we have the possibility that the last bit added creates a forbidden 00.

The formula turns out to be related to $F(n)$, and it has the same annoying property $F(n)$ using the definition requires about $n$ calculations.
The course notes present a proof by induction that

\[ F(n) = \frac{\phi^n - (\hat{\phi})^n}{\sqrt{5}}, \quad \phi = \frac{1 + \sqrt{5}}{2}, \hat{\phi} = \frac{1 - \sqrt{5}}{2} \]

You can, and should, be able to work through the proof. The question remains, why did somebody think of $\phi$ and $\hat{\phi}$?
Start with the idea that $F(n)$ seems to increase by something close to a fixed ratio. Let’s try calling that $r$, and $r$ has to satisfy:

$$r^n = r^{n-1} + r^{n-2} \Rightarrow r^2 = r + 1$$

There are two solutions to the quadratic equation: $\phi$ and $\hat{\phi}$, but what about the $1/\sqrt{5}$ factor?

If $\phi$ and $\hat{\phi}$ are solutions, so are linear combinations:

$$\alpha \phi^n + \beta \hat{\phi}^n = \alpha \phi^{n-1} + \beta \hat{\phi}^{n-1} + \alpha \phi^{n-2} + \beta \hat{\phi}^{n-2}$$
Rabbits, hats

Match up $\alpha$ and $\beta$ to solutions:

\[ \alpha \phi^0 + \beta \hat{\phi}^0 = 0 \quad \Rightarrow \quad \alpha = -\beta \]

\[ \alpha \phi^1 + \beta \hat{\phi}^1 = 1 \quad \Rightarrow \quad \alpha (\phi - \hat{\phi}) = 1 \]
Notes