Using Introduction to the Theory of Computation, Section 1.3
Outline

principle of complete induction

eamples of complete induction

well-ordering
Complete Induction
another flavour needed

Every natural number greater than 1 has a prime factorization

Try some examples

How does the factorization of 8 help with the factorization of 9?
More dominoes

\[
(\forall n \in \mathbb{N}, [\forall k \in \{k \in \mathbb{N} \mid k < n\}, P(k)] \Rightarrow P(n)) \Rightarrow \forall n \in \mathbb{N}, P(n)
\]

If all the previous cases always implies the current case then all cases are true
Every natural number greater than 1 has a prime factorization

\( \forall n \in \mathbb{N}, [\forall k \in \{k \in \mathbb{N} \mid k < n\}, P(k)] \Rightarrow P(n) \Rightarrow \forall n \in \mathbb{N}, P(n) \)
Every natural number greater than 1 has a prime factorization

$$(\forall n \in \mathbb{N}, [\forall k \in \{k \in \mathbb{N} \mid k < n\}, P(k)] \Rightarrow P(n)) \Rightarrow \forall n \in \mathbb{N}, P(n)$$
A tree is a directed graph.

A non-empty tree has a root node, $r$, such that there is exactly one path from $r$ to any other node.

If a tree has an edge $(u, v)$, then $u$ is $v$’s parent, $v$ is $u$’s child.

Two nodes with the same parent are called siblings.

A node with no children is called a leaf. A non-leaf is called an internal node.

Binary trees have nodes with $\leq 2$ children, and children are labelled either left or right.

Internal nodes of full binary trees have 2 children.
Tree examples

know your trees...
Every full binary tree, except the zero tree, has an odd number of nodes

\((\forall n \in \mathbb{N}, [\forall k \in \{k \in \mathbb{N} \mid k < n\}, P(k)] \Rightarrow P(n)) \Rightarrow \forall n \in \mathbb{N}, P(n)\)
Every full binary tree, except the zero tree, has an odd number of nodes

\((\forall n \in \mathbb{N}, [\forall k \in \{k \in \mathbb{N} \mid k < n\}, P(k)] \Rightarrow P(n)) \Rightarrow \forall n \in \mathbb{N}, P(n)\)
Well-ordering example

\( \forall n, m \in \mathbb{N}, n \neq 0, \ R = \{ r \in \mathbb{N} | \exists q \in \mathbb{N}, m = qn + r \} \) has a smallest element

This is the main part of proving the existence of a unique quotient and remainder:

\[ \forall m \in \mathbb{N}, \forall n \in \mathbb{N} - \{0\}, \exists q, r \in \mathbb{N}, m = qn + r \land 0 \leq r < n \]

The course notes use Mathematical Induction. Well-ordering is shorter and clearer.
Principle of well-ordering

Every non-empty subset of \( \mathbb{N} \) has a smallest element

Is there something similar for \( \mathbb{Q} \) or \( \mathbb{R} \)?

For a given pair of natural numbers \( m, n \neq 0 \) does the set \( R \) satisfy the conditions for well-ordering?

\[
R = \{ r \in \mathbb{N} \mid \exists q \in \mathbb{N}, m = qn + r \}
\]

If so, we still need to be sure that

1. \( 0 \leq r < n \)
2. That \( q \) and \( r \) are unique — no other natural numbers would work

in order to have

\[
\forall m \in \mathbb{N}, \forall n \in \mathbb{N} - \{0\}, \exists q, r \in \mathbb{N}, m = qn + r \land 0 \leq r < n
\]
Every non-empty subset of \( \mathbb{N} \) has a smallest element

\[ \forall m \in \mathbb{N}, \forall n \in \mathbb{N} - \{0\}, \exists q, r \in \mathbb{N}, m = qn + r \land 0 \leq r < n \]
Every non-empty subset of $\mathbb{N}$ has a smallest element

$\forall m \in \mathbb{N}, \forall n \in \mathbb{N} - \{0\}, \exists q, r \in \mathbb{N}, m = qn + r \land 0 \leq r < n$