UNIVERSITY OF TORONTO
Faculty of Arts and Science

TERM TEST #1
CSC 236H1
Duration — 50 minutes

LAST NAME: _______________________________________
FIRST NAME: _____________________________________

Do not turn this page until you have received the signal to start.
(In the meantime, please fill out the identification section above, and read the instructions below.)

This test consists of 3 questions on 7 pages (including this one). When you receive the signal to start, please make sure that your copy of the test is complete.

Please answer questions in the space provided. You will earn 20% for any question you leave blank or write “I cannot answer this question,” on. You will earn substantial part marks for writing down the outline of a solution and indicating which steps are missing.

Good Luck!

Total Pages = 7
QUESTION 1. [5 marks]

Use Mathematical Induction to prove that for every natural number \( n \), \( 13^n - 1 \) is a multiple of 12.

**Proof:** For convenience I define \( P(n) : \exists z \in \mathbb{Z}, 13^n - 1 = 12z \) I use Mathematical Induction to prove \( \forall n \in \mathbb{N}, P(n) \).

Base case, \( n=0 \): If \( n = 0 \) then \( 13^0 - 1 = 13^0 - 1 = 0 \) and \( 0 = 12 \times 0, 0 \in \mathbb{Z} \), so \( P(0) \) is verified.

Induction step: Assume \( n \in \mathbb{N} \) and \( P(n) \). Then

\[
13^{n+1} - 1 = 13(13^n - 1) + 12 \quad \text{# rewrite}
\]
\[
= 13(12z) + 12, \text{ some } z \in \mathbb{Z} \quad \text{# by } P(n)
\]
\[
= 12(13z + 1) = 12z', z' \in \mathbb{Z} \quad \text{# } 13, z, 1 \in \mathbb{Z}
\]
\[
\text{# and } \mathbb{Z} \text{ is closed under } +, \times
\]

Since I assume \( n \) was an arbitrary natural number and \( P(n) \), and from that derived \( P(n + 1) \), then \( \forall n \in \mathbb{N}, P(n) \Rightarrow P(n + 1) \).

I conclude, by mathematical induction, \( \forall n \in \mathbb{N}, P(n) \).
Recall the definition of a binary tree (a tree where each node has, at most, two children). Define the height of a binary tree as the number of nodes in the longest path from root to a leaf — so a tree consisting of a single node has height 1. Use Complete Induction to prove that for every natural number \( n \), any non-empty binary tree of height \( n \) has, at most, \( 2^{n-1} \) leaves (leaves are nodes with zero children).

SOLUTION: For convenience, I define the predicate \( P(n) : \) "Any non-empty binary tree of height \( n \) has, at most \( 2^{n-1} \) leaves." I use complete induction to prove \( \forall n \in \mathbb{N}, P(n) \).

Induction Step: Assume \( n \in \mathbb{N} \), and that \( P(i) \) is true for all \( 0 \leq i < n \).

Case \( n \leq 1 \): The only non-empty binary tree with height no greater than 1 consists of a childless root, and has height 1. Since the root is also the sole leaf, this tree has no more than \( 2^{1-1} = 1 \) leaf, so \( P(1) \) is verified (and \( P(0) \) is vacuously verified, since there are no non-empty binary trees of height 0).

Case \( n > 1 \): A tree with height 2 or more has either 1 or two non-empty sub-trees, and these contribute all the leaves. I consider the cases separately.

Sub-case A, exactly one non-empty sub-tree: In this case, the single sub-tree has height \( n - 1 \) (the longest path, minus the root node), and by the IH it has no more than \( 2^{n-1-1} \) leaves, which (in turn) is no more than \( 2^{n-1} \) leaves.

Sub-case B, exactly two non-empty sub-trees: Call the height of the left sub-tree \( n_L \), and that of the right sub-tree \( n_R \). Since they are non-empty, \( 1 \leq n_L, n_R \), and since the longest path in each sub-tree is shorter (by one) than the corresponding path that includes the root of the original tree, they have \( n_L, n_R \leq n - 1 \). This means that the IH applies, and the total number of leaves contributed by the two sub-trees is

\[
2^{n_L-1} + 2^{n_R-1} \leq 2^{n-1-1} + 2^{n-1-1} = 2^{n-1}
\]

So \( \forall n \in \mathbb{N} \), if \( P(i) \) for \( 0 \leq i < n \), then \( P(n) \).

Then, by complete induction, I conclude \( \forall n \in \mathbb{N}, P(n) \).

(Another approach would be to strengthen the claim so that it applied to all binary trees (not just non-empty ones), and the claim about non-empty binary trees would follow as a consequence).
QUESTION 3.  [5 marks]

Define $E$ as the smallest set such that

1. Symbols $x$, $y$, and $z$ are elements of $E$.

2. If expressions $e_1$ and $e_2$ are elements of $E$, then so is $(e_1 + e_2)$.

For $e \in E$ define $\text{vr}(e)$ as the number of occurrences of symbols from the set \{x, y, z\} in $e$, and $\text{p}(e)$ as the number of occurrences of parentheses from the set \{(, )\} in $e$. For example, $\text{vr}(((x + y) + z)) = 3$ and $\text{p}(((x + y) + z)) = 4$. Use Structural Induction to prove that for every $e \in E$, $2\text{vr}(e) = \text{p}(e) + 2$.

SOLUION: for convenience, I define the predicate $P(e) \equiv "2\text{vr}(e) = \text{p}(e) + 2"$ I use structural induction to prove $\forall e \in E, P(e)$.

Basis, $e \in \{x, y, z\}$: Each of these expressions has exactly one variable and 0 parentheses, and $2 \times 1 = 0 + 2$, so $P(e)$ is verified for $e$ in the basis.

Induction Step: Assume $e_1, e_2 \in E$ and $P(e_1), P(e_2)$. Then

$$2\text{vr}((e_1 + e_2)) = 2(\text{vr}(e_1) + \text{vr}(e_2)) \quad \text{# no variables added}$$

$$= 2\text{vr}(e_1) + 2\text{vr}(e_2)$$

$$= \text{p}(e_1) + 2 + \text{p}(e_2) + 2 = [\text{p}(e_1) + \text{p}(e_2) + 2] + 2 \quad \text{# by IH}$$

$$= \text{p}((e_1 + e_2)) + 2 \quad \text{# 2 parentheses needed to wrap } (e_1 + e_2)$$

So, $\forall e_1, e_2 \in E, P(e_1) \land P(e_2) \Rightarrow P((e_1 + e_2))$.

By the principle of structural induction, I conclude $\forall e \in E, P(e)$. 
# 1: _____/ 5
# 2: _____/ 5
# 3: _____/ 5

TOTAL: _____/15

End of Exam