CSC236 tutorial exercises #4
(Best before 11 am, Monday October 22nd)

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These exercises are meant to give you practice with some of the concepts used to prove the Master Theorem.

1. Consider the recurrence:

\[ T(n) = \begin{cases} 
1 & \text{if } n = 1 \\
T\left(\left\lfloor \frac{n}{2} \right\rfloor \right) + T\left(\left\lfloor \frac{n}{2} \right\rfloor \right) + n + 1 & \text{if } n > 1
\end{cases} \]

This recurrence is superficially different from the one derived in the Course notes. Use the above recurrence and the approach of Lemma 3.7 in the Course Notes to show that \( T \) is non-decreasing.

**Claim:** Define \( P(n) : \) for every positive integer \( m, m < n \Rightarrow T(m) \leq T(n) \). I use complete induction to prove that \( \forall n \in \mathbb{N}^+, P(n) \).

**Induction step:** Assume that \( n \) is an arbitrary positive integer, and that \( P(k) \) is true for \( 1 \leq k < n \).

**Case 1 \leq n < 3:** \( P(1) \) is vacuously true, since there are no positive integers less than 1. To establish \( P(2) \) I calculate \( T(1) = 1 \) and \( T(2) = 2T(1) + 2 + 1 = 5 \), and note that \( 1 \leq 5 \), so \( P(1) \) and \( P(2) \) are each verified.

**Case \( n > 2 \):** Then \( 1 \leq \left\lfloor \frac{n}{2} \right\rfloor \leq \left\lfloor \frac{n}{2} \right\rfloor < n \), by Lemma 3.8. Also \( 1 \leq n - 1 < n \), so \( P(n - 1) \) is true, by assumption, and the only thing left is to show \( T(n - 1) \leq T(n) \), and

\[
T(n - 1) = T\left(\left\lfloor \frac{(n - 1)}{2} \right\rfloor \right) + T\left(\left\lfloor \frac{(n - 1)}{2} \right\rfloor \right) + (n - 1) + 1 \quad \# \text{ apply definition} \\
\leq T\left(\left\lfloor \frac{n}{2} \right\rfloor \right) + T\left(\left\lfloor \frac{n}{2} \right\rfloor \right) + n + 1 \quad \# \text{ by } P\left(\left\lfloor \frac{n}{2} \right\rfloor \right), P\left(\left\lfloor \frac{n}{2} \right\rfloor \right) \\
\quad \# \text{ also } n - 1 \leq n \text{ and } 1 \leq \left\lfloor \frac{(n - 1)}{2} \right\rfloor \leq \left\lfloor \frac{n}{2} \right\rfloor \\
= T(n)
\]

Conclude \( \forall n \in \mathbb{N}^+, P(n) \) by complete induction.
2. Use repeated substitution (unwinding) to find a closed form for the recurrence $S$ when $n$ is a power of 3:

$$S(n) = \begin{cases} 
1 & \text{if } n < 3 \\
 a_1 S([n/3]) + a_2 S([n/3]) + n^2 & \text{if } n > 2
\end{cases}$$

... where integers $a_1, a_2 \geq 0$ and $a_1 + a_2 = 3$.

**Solution:** If $n$ is an integer power of 3 greater than $3^0$, then $[n/3]$ is the same as $[n/3]$, and the recurrence can be simplified to:

$$S(n) = 3S(n/3) + n^2$$

Unwind this a few steps to see a pattern:

$$S(n) = 3S(n/3) + n^2$$

$$S(n) = 3(3S(n/9) + (n/3)^2) + n^2$$

$$S(n) = 3^2 S(n/9) + n^2/3 + n^2$$

$$S(n) = 3^2 (3S(n/27) + (n/9)^2) + n^2/3 + n^2$$

$$S(n) = 3^2 S(n/27) + n^2/9 + n^2/3 + n^2$$

$$\vdots$$

$$S(n) = 3^n S(n/n) + n^2 \sum_{i=0}^{k-1} 1/3^i \quad \# k = \log_3 n$$

$$S(n) = n + n^2 \frac{1 - (1/3)^k}{1 - 1/3} \quad \# \text{formula for geometric series}$$

$$S(n) = n + n^2 \frac{3^k - 1}{2} \frac{1}{3 \cdot 2} = n + 3n^2(n-1) \quad \# n = 3^n$$

$$S(n) = n + 3^2 n(n-1)$$

Claim: $\forall k \in \mathbb{N}, S(3^k) = 3^k + \frac{3}{2} 3^k (3^k - 1)$. For convenience, define $P(k) : S(3^k) = 3^k + \frac{3}{2} 3^k (3^k - 1)$.

**Proof (by mathematical induction):**

**Base case $k = 0$:** By definition $S(3^0) = 1$, and that’s also equal to $3^0 + \frac{3}{2} 3^0 (3^0 - 1)$, as claimed. So the $P(0)$ is true.

**Induction step:** Assume that $k$ is an arbitrary natural number and assume that $P(k)$ is true. Then

$$S(3^{k+1}) = 3S(3^k) + (3^{k+1})^2 \quad \# \text{apply definition for } 3^{k+1} > 2$$

$$S(3^{k+1}) = 3 \left( 3^k + \frac{3}{2} 3^k (3^k - 1) \right) + (3^{k+1})^2 \quad \# \text{apply IH to } S(3^k)$$

$$S(3^{k+1}) = 3^{k+1} + \frac{3}{2} 3^{k+1} (3^{k-1}) + (3^{k+1})^2$$

$$S(3^{k+1}) = 3^{k+1} + 3^{k+1} \frac{3^{k+1} - 3 + 2 \times 3^{k+1}}{2} \quad \# \text{factor out } 3^{k+1}$$

$$S(3^{k+1}) = 3^{k+1} + \frac{3}{2} 3^{k+1} (3^{k+1} - 1)$$

So $S(3^{k+1}) = 3^{k+1} + \frac{3}{2} 3^{k+1} (3^{k+1} - 1)$. So, $\forall k \in \mathbb{N}, P(k) \Rightarrow P(k+1)$.

Conclude, $\forall k \in \mathbb{N}, P(k)$, by mathematical induction.