CSC236 fall 2012
recursion, induction, correctness

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Using Introduction to the Theory of Computation,
Chapter 2
Outline

binary search
recursive binary search

def recBinSearch(x, A, b, e):
1.    if b == e:
2.        if x \leq A[b]:
3.            return b
4.        else:
5.            return e + 1
6.    else:
7.        m = (b + e) // 2  # midpoint
8.        if x \leq A[m]:
9.            return recBinSearch(x, A, b, m)
10.       else:
11.          return recBinSearch(x, A, m+1, e)
conditions, pre- and post-

- $x$ and elements of $A$ are comparable
- $e$ and $b$ are valid indices, $b \leq e$
- $A[b..e]$ is sorted non-decreasing

RecBinSearch($x, A, b, e$) terminates and returns index $p$

- $b \leq p \leq e + 1$
- $b < p \implies A[p - 1] < x$
- $p \leq e \implies x \leq A[p]$

(except for boundaries, returns $p$ so that $A[p - 1] < x \leq A[p]$)
precondition ⇒ termination and postcondition

Proof: induction on \( n = e - b + 1 \)

Base case, \( n = 1 \): Terminates because there are no loops or further calls, returns \( x \leq A[b = p] \iff p = b = e \) is returned. \( x > A[b = p - 1] \iff p = b + 1 \) returned, so postcondition satisfied. Notice that the choice forces if-and-only-if.

Induction step: Assume \( n > 1 \) and that the postcondition is satisfied for inputs of size \( 1 \leq k < n \) that satisfy the precondition. Call RecBinSearch(A,x,b,e) when \( n = e - b + 1 > 1 \). Since \( b < e \) in this case, the test on line 1 fails, and line 7 executes. Exercise: \( b \leq m < e \) in this case. There are two cases, according to whether \( x \leq A[m] \) or \( x > A[m] \).
Case 1: $x \leq A[m]$

Show that IH applies to $RBS(x, A, b, m)$

Translate the postcondition to $RBS(x, A, b, m)$

- $b \leq p \leq m + 1$
- $b < p \Rightarrow A[p-1] \leq x$
- $p \leq m \Rightarrow x \leq A[p]$

Show that $RBS(x, A, b, e)$ satisfies postcondition

- $b \leq p \leq m + 1 \leq e \leq e + 1$
- $b < p \Rightarrow A[p-1] \leq x$
- $p \leq e \Rightarrow x \leq A[p]$ must show $x \leq A[p]$
Case 2: \( x > A[m] \)

- Show that IH applies to \( \text{RBS}(x,A,m+1,e) \)
- Translate postcondition to \( \text{RBS}(x,A,m+1,e) \)
  - \( b < m+1 \leq p \leq e+1 \)
  - \( p \leq e \Rightarrow x \leq A[p] \)
  - \( m+1 < p \Rightarrow A[p-1] < x \)

- Show that \( \text{RBS}(x,A,b,e) \)
  - \( b \leq p \leq e+1 \) \( \text{by IH and } m \geq b \)
  - \( p \leq e \Rightarrow x \leq A[p] \) \( \text{by IH} \)
  - Show \( b < p \Rightarrow A[p-1] < x \)
  - \( p > m+1 \), then \( A[p-1] < x \) \( \text{by IH} \)
  - \( p \leq m+1 \Rightarrow p = m+1 \), then \( x > A[m] = A[p-1] \)
what could go wrong?

\[
m = \left\lfloor \frac{e+1}{2.0} \right\rfloor
\]

- \( m = \left\lfloor \frac{e+b}{2.0} \right\rfloor \)

- \( x < A[m] \) — prove or counterexample

- ... 

- Either prove correct, or find a counter-example