TD papers - after (some) FSAs
A3 - up tonight.

CSC236 fall 2012
automata and languages

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Using Introduction to the Theory of Computation,
Chapter 7
Outline

formal languages

FSAs

notes
some definitions

**alphabet:** finite, non-empty set of symbols, e.g. \{a, b\} or \{0, 1, −1\}. Conventionally denoted \(\Sigma\).

**string:** finite (including empty) sequence of symbols over an alphabet: abba is a string over \{a, b\}. Convention: \(\varepsilon\) is the empty string, never an allowed symbol, \(\Sigma^*\) is set of all strings over \(\Sigma\).

**language:** Subset of \(\Sigma^*\) for some alphabet \(\Sigma\). Possibly empty, possibly infinite subset. E.g. \{\}, \{aa, aaa, aaaa, ...\}.

**N.B.:** \{\} \(\neq\) \{\(\varepsilon\)\}. 


Many problems can be reduced to languages: logical formulas, identifiers for compilation, natural language processing. Key question is recognition:

Given language $L$ and string $s$, is $s \in L$?

Languages may be described either by descriptive generators (for example, regular expressions) or procedurally (e.g. finite state automata)
more notation

**string length:** denoted $|s|$, is the number of symbols in $s$, e.g. $|bba| = 3$.

$s = t$: if and only if $|s| = |t|$, and $s_i = t_i$ for $1 \leq i \leq |s|$.

$s^R$: reversal of $s$ is obtained by reversing symbols of $s$, e.g. $1011^R = 1101$.

$s t$ or $s \circ t$: concatenation of $s$ and $t$ — all characters of $s$ followed by all those of $t$, e.g. $bba \circ bb = bbabb$.

$s^k$: denotes $s$ concatenated with itself $k$ times. E.g., $ab^3 = ababab$, $101^0 = \varepsilon$.

$\Sigma^n$: all strings of length $n$ over $\Sigma$, $\Sigma^*$ denotes all strings over $\Sigma$. 
language operations

$\bar{L}$: Complement of $L$, i.e. $\Sigma^* - L$. If $L$ is language of strings over $\{0, 1\}$ that start with 0, then $\bar{L}$ is the language of strings that begin with 1 plus the empty string.

$L \cup L'$: union

$L \cap L'$: intersection

$L - L'$: difference

$\text{Rev}(L)$: $= \{s^R : s \in L\}$

concatenation: $LL'$ or $L \cdot L' = \{rt : r \in L, t \in L'\}$. Special cases

$L\{\epsilon\} = L = \{\epsilon\}L$, and $L\{\} = \{\} = \{\}L$. 
more language operations

exponentiation: $L^k$ is concatenation of $L$ $k$ times. Special case, $L^0 = \{\varepsilon\}$, including $L = \{\}\}$!

Kleene star: $L^* = L^0 \cup L^1 \cup L^2 \cup \ldots$
states needed to classify a string
what state is a stingy vending machine in based on coins?
accepts only nickles (a), dimes (b), and quarters (c),
no change given, and everything costs 30 cents
useful toy (you’ll need JRE)

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build an automaton with formalities...

quintuple: \((Q, \Sigma, q_0, F, \delta)\)

- \(Q\) is set of states, \(\Sigma\) is finite, non-empty alphabet, \(q_0\) is start state
- \(F\) is set of accepting states, and \(\delta : Q \times \Sigma \rightarrow Q\) is transition function

We can extend \(\delta : Q \times \Sigma \rightarrow Q\) to a transition function that tells us what state a string \(s\) takes the automaton to:

\[
\delta^* : Q \times \Sigma^* \rightarrow Q \\
\delta^*(q, s) = \begin{cases} 
q & \text{if } s = \varepsilon \\
\delta(\delta^*(q, s'), a) & \text{if } s' \in \Sigma^*, a \in \Sigma, s = \end{cases}
\]

String \(s\) is accepted if and only iff \(\delta^*(q_0, s) \in F\), it is rejected otherwise.
example — an odd machine
devise a machine that accepts strings over \( \{a, b\} \) with an odd number of \( a \)s

Formal proof requires inductive proof of invariant:

\[
\delta^*(E, s) = \begin{cases} 
E & \text{if } s \text{ has even number of } a \text{s} \\
O & \text{if } s \text{ has odd number of } a \text{s}
\end{cases}
\]

induction on \(|S|\)
structural induction on \(S\)
\(\Sigma = \{a, b\}^3\)

Def \(\Sigma^*\): smallest set such that

1. \(\emptyset \in \Sigma^*\)
2. (induction step) if \(y' \in \Sigma^*\), then \(y'c \in \Sigma^*\), for \(c \in \Sigma\)
float machine

$L_1 = \{0, \ldots, 9\}$
$L_2 = \{+, -\}, L_3 = \{.\}$
$L_F = \{s \in L_2 L_1^j L_3^k L_1^n | j, k \leq 1, m, n \geq 1\}$

Devise a machine that accepts $L_F$
more odd/even

$L$ is the language of binary strings
with an odd number of $a$s, but even length
Devise a machine for $L$
notes