1. Proving equivalence

Suppose $P$, $Q$, $R$, and $S$ are statements.

1. Prove that $P \Rightarrow (Q \Rightarrow (R \Rightarrow S))$ is equivalent to $(P \land Q \land R) \Rightarrow S$.
2. Prove that $((P \Rightarrow Q) \Rightarrow R) \Rightarrow S$ is equivalent to $(\neg P \land \neg R) \lor (Q \land \neg R) \lor S$.

2. Negation

Negate the following sentences:

1. Every dog has its day, or perhaps its cat.
2. $\forall x \in X, \exists y \in Y, x > y \land y > x$

3. Guarantees

Consider the statement:

$(S1)$ A and B are both guarantees that C is true.

1. Write $(S1)$ symbolically. Use parentheses “(” and “)” to make your answer precise.
2. Choose some appropriate phrases to replace A, B and C. Use these to write $(S1)$ in English. Does this cause you to reconsider your answer to (1)?
3. Suppose $(S1)$ is true and A is false. What, if anything, can be determined about B and C? Briefly justify.
4. Sample solutions

1. Proving Equivalence

(1) Prove that \( P \Rightarrow (Q \Rightarrow (R \Rightarrow S)) \) is equivalent to \( (P \land Q \land R) \Rightarrow S \).

\[
P \Rightarrow (Q \Rightarrow (R \Rightarrow S)) \iff \neg P \lor (-Q \lor (-R \lor S)) \quad \text{[transform \( \Rightarrow \) to \( \neg \) and \( \lor \])}
\]
\[
\iff \neg P \lor (-Q \lor (-R \lor S)) \quad \text{[associativity of \( \lor \)]}
\]
\[
\iff \neg P \lor (-Q \lor (-R \lor S)) \quad \text{[DeMorgan's Law]}
\]
\[
\iff (P \land Q \land R) \Rightarrow S \quad \text{[transform \( \neg \) and \( \lor \) to \( \Rightarrow \)].}
\]

(2) Prove that \(((P \Rightarrow Q) \Rightarrow R) \Rightarrow S\) is equivalent to \( (-P \land \neg R) \lor (Q \land \neg R) \lor S \).

\[
((P \Rightarrow Q) \Rightarrow R) \Rightarrow S \iff \neg((P \lor Q) \lor R) \lor S \quad \text{[transform \( \Rightarrow \) to \( \neg \) and \( \lor \])}
\]
\[
\iff ((P \lor Q) \land \neg R) \lor S \quad \text{[DeMorgan's Law]}
\]
\[
\iff (-P \land \neg R) \lor (Q \land \neg R) \lor S \quad \text{[distributivity of \( \land \)]}
\]

5. Negation

(1) Every dog has its day, or perhaps its cat.

Sample solution: Some dog has neither its day nor its cat.

(2) \( \forall x \in X, \exists y \in Y, x > y \land y > x \)

Sample solution: \( \exists x \in X, \forall y \in Y, x \leq y \lor y \leq x \)

2. Guarantees

(1) \((A \Rightarrow C) \land (B \Rightarrow C)\) or \((A \lor B) \Rightarrow C \)

(2) “Being rich and being beautiful are both guarantees that one is hated.”

(3) Suppose \((S1)\) is true and \(A\) is false. What, if anything, can be determined about \(B\) and \(C\)? Briefly justify.

Nothing. It tells us nothing about \(C\), and \(A\) is unrelated to \(B\).

(4) Suppose \((S1)\) is true and \(C\) is false. What, if anything, can be determined about \(A\) and \(B\)? Briefly justify.

\(A\) is false and \(B\) is false. This comes from the contrapositive(s) of the implication(s), which must be true.