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Mathematical expression

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Course notes, chapter 3
non-boolean functions

notes
non-boolean functions

Take care when expressing a proof about a function that returns a non-boolean value, such as a number:

\[ \lfloor x \rfloor \text{ is the largest integer } \leq x. \]

Now prove the following statement (notice that we quantify over \( x \in \mathbb{R} \), not \( \lfloor x \rfloor \in \mathbb{R} \)):

\[ \forall x \in \mathbb{R}, \lfloor x \rfloor < x + 1 \]
You may have been disappointed that the last proof used only part of the definition of floor. Here’s a symbolic re-writing of the definition of floor:

\[ \forall x \in \mathbb{R} \quad y = \lfloor x \rfloor \iff y \in \mathbb{Z} \land y \leq x \land (\forall z \in \mathbb{Z}, z \leq x \implies z \leq y) \]

The full version of the definition should prove useful to prove:

\[ \forall x \in \mathbb{R}, \lfloor x \rfloor > x - 1 \]
proving something false

Define a sequence:

$$\forall n \in \mathbb{N} \quad a_n = \lfloor n/2 \rfloor$$

(of course, if you treat "/" as integer division, there’s no need to take the floor. Now consider the claim:

$$\exists i \in \mathbb{N}, \forall j \in \mathbb{N}, j > i \Rightarrow a_j = a_i$$

The claim is false. Disprove it.
Sometimes your argument has to split to take into account possible properties of your generic element:

\[ \forall n \in \mathbb{N}, n^2 + n \text{ is even} \]

A natural approach is to factor \( n^2 + n \) as \( n(n + 1) \), and then consider the case where \( n \) is odd, then the case where \( n \) is even.
proof about limits

In proving this claim you can’t control the value of $\varepsilon$ or $y$, but you can craft $\delta$ to make things work out.

$$\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+, \forall y \in \mathbb{R}, |y - \pi| < \delta \Rightarrow |y^2 - \pi^2| < \varepsilon$$

The claim is true. The proof format should be already familiar to you. A good approach is to fill in as much as possible, leaving the actual value of $\delta$ out until you have more intuition about it.
Suppose you have a predicate of the natural numbers:

$$\forall n \in \mathbb{N} \quad S(n) \iff \exists k \in \mathbb{N}, n = 7k + 3$$

Is $S(3 \times 3)$ true? How do you prove that? It’s useful to check out the remainder theorem from the sheet of mathematical prerequisites.
Be careful proving a claim false. Consider the claim, for some suitably defined $X$, $Y$ and $P$, $Q$:

$$S : \quad \forall x \in X, \forall y \in Y, P(x, y) \Rightarrow Q(x, y)$$

To disprove $S$, should you prove:

$$\forall x \in X, \forall y \in Y, P(x, y) \Rightarrow \neg Q(x, y)$$

What about

$$\forall x \in X, \forall y \in Y, \neg (P(x, y) \Rightarrow Q(x, y))$$

Explain why, or why not.
Define $T(n)$ by:

$$\forall n \in \mathbb{N} \quad T(n) \iff \exists i \in \mathbb{N}, n = 7i + 1.$$ 

Take some scrap paper, don’t write your name on it, and fill in as much of the proof of the following claim as possible:

$$\forall n \in \mathbb{N}, T(n) \Rightarrow T(n^2)$$

Now fill in as much of the disproof of the following claim as possible:

$$\forall n \in \mathbb{N}, T(n^2) \Rightarrow T(n)$$