CSC165 winter 2013
Mathematical expression

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Course notes, chapter 3
Outline

universally quantified implication, cont’d

existence

notes
proof outline

More flexible format required in this course. Each link in the chain justified by mentioning supporting evidence in a comment beside it. Here are portions of an argument where scope of assumption is shown by indentation. A generic proof that \( \forall x \in X, P(x) \Rightarrow Q(x) \) might look like:

Assume \( x \in X \) \# \( x \) is generic; what I prove applies to all of \( X \)

Assume \( P(x) \). \# Antecedent. Otherwise, \( \neg P(x) \) means we get the implication for free.

Then \( R_1(x) \) \# by previous result

\( C2.0, \forall x \in X, P(x) \Rightarrow R_1(x) \)
Then \( R_2(x) \) \# by previous result

\( C2.1, \forall x \in X, R_1(x) \Rightarrow R_2(x) \)

\[ \vdots \]

Then \( Q(x) \) \# by previous result

\( C2.n, \forall x \in X, R_n(x) \Rightarrow Q(x) \)

Then \( P(x) \Rightarrow Q(x) \) \# I assumed antecedent, got consequent (aka introduced \( \Rightarrow \))

Then \( \forall x \in X, P(x) \Rightarrow Q(x) \) \# reasoning works for all \( x \in X \)
Prove that for every pair of non-negative real numbers \((x, y)\), if \(x\) is greater than \(y\), then the geometric mean, \(\sqrt{xy}\) is less than the arithmetic mean, \((x + y)/2\).
Prove that for any natural number $n$, $n^2$ odd implies that $n$ is odd.
proving existence

To prove the a set is non-empty, it’s enough to exhibit one element. How do you prove:

$$\exists x \in \mathbb{R}, x^3 + 3x^2 - 4x = 12$$
prove a claim about a sequence

Define sequence $a_n$ by:

$$\forall n \in \mathbb{N} \quad a_n = n^2$$

Now prove:

$$\exists i \in \mathbb{N}, \forall j \in \mathbb{N}, a_j \leq i \Rightarrow j < i$$
contradiction — a special case of contrapositive

Define the prime natural numbers as
\[ P = \{ p \in \mathbb{N} \mid p \text{ has exactly two distinct divisors in } \mathbb{N} \} \]. How do you prove:

\[ S : \quad \forall n \in \mathbb{N}, |P| > n \]

It would be nice to have some result \( R \) that leads to \( S \). If you could show \( R \Rightarrow S \), and that \( R \) is true, then you’d be done. But, out of many elementary results, how do you choose an \( R \)? Contradiction will often lead you there.
non-boolean functions

Take care when expressing a proof about a function that returns a non-boolean value, such as a number:

\[ \lfloor x \rfloor \text{ is the largest integer } \leq x. \]

Now prove the following statement (notice that we quantify over \( x \in \mathbb{R} \), not \( \lfloor x \rfloor \in \mathbb{R} \):

\[ \forall x \in \mathbb{R}, \lfloor x \rfloor < x + 1 \]
You may have been disappointed that the last proof used only part of the definition of floor. Here’s a symbolic re-writing of the definition of floor:

$$\forall x \in \mathbb{R} \quad y = \lfloor x \rfloor \leftrightarrow y \in \mathbb{Z} \land y \leq x \land (\forall z \in \mathbb{Z}, z \leq x \Rightarrow z \leq y)$$

The full version of the definition should prove useful to prove:

$$\forall x \in \mathbb{R}, \lfloor x \rfloor > x - 1$$
proving something false

Define a sequence:

$$\forall n \in \mathbb{N} \quad a_n = \lfloor n/2 \rfloor$$

(of course, if you treat “/” as integer division, there’s no need to take the floor. Now consider the claim:

$$\exists i \in \mathbb{N}, \forall j \in \mathbb{N}, j > i \Rightarrow a_j = a_i$$

The claim is false. Disprove it.
proof by cases

Sometimes your argument has to split to take into account possible properties of your generic element:

\[ \forall n \in \mathbb{N}, n^2 + n \text{ is even} \]

A natural approach is to factor \( n^2 + n \) as \( n(n + 1) \), and then consider the case where \( n \) is odd, then the case where \( n \) is even.