CSC165 winter 2013
Mathematical expression

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Course notes, chapter 2–3
implication as disjunction

mixed quantifiers

Notes
implication two ways

The result of the following truth table is useful enough to bear restating:

\[
\begin{array}{c|c|c|c}
P & Q & P \Rightarrow Q & \neg P \lor Q \\
\hline
T & T & T & T \\
T & F & F & T \\
F & T & T & T \\
F & F & T & T \\
\end{array}
\]
Translate bi-implication into the conjunction of two disjunctions:

\[ (P \Rightarrow Q) \land (Q \Rightarrow P) \]

Now change your expression for bi-implication into the disjunction of two conjunctions (use the some of the equivalences from a few slides ago):

\[ \equiv \left( \neg P \lor \neg Q \right) \lor \left( \neg Q \lor \neg P \right) \equiv \neg \neg P \land \neg \neg Q \lor \neg Q \land \neg P \equiv \neg P \land \neg Q \lor Q \land P \]

What’s the negation of bi-implication? How would you explain it in English?

\[ P \land \neg Q \lor Q \land \neg P \rightarrow \Theta \text{ XOR OR} \]
transitivity

What does the following statement mean, when you interpret it as a venn diagram?

$$\forall x \in X, (P(x) \Rightarrow Q(x)) \land (Q(x) \Rightarrow R(x))$$

For another insight, negate the following statement, and simplify it by transforming implications into disjunctions:

$$\neg [(P \Rightarrow Q) \land (Q \Rightarrow R)] \Rightarrow (P \Rightarrow R)$$

$$= \neg [(P \Rightarrow Q) \land (Q \Rightarrow R)] \land P \land \neg R$$

$$= [(P \lor Q) \land (\neg Q \lor R)] \land P \land \neg R$$
for all, one...one for all

What's the difference between these two claims:

\[ \forall x \in L_1, \exists y \in L_2, x + y = 5 \]
\[ \exists y \in L_2, \forall x \in L_1, x + y = 5 \]

```python
def P(x, y):
    return x + y == 5
L1 = L2 = [1, 2, 3, 4]

def forallExists(P, L1, L2):
    return False not in [True in [P(x, y) for y in L2] for x in L1]

def existsForall(P, L1, L2):
    return True in [False not in [P(x, y) for x in L2] for y in L1]
```
Can you switch \( \forall \varepsilon \in \mathbb{R}^+ \) with \( \exists \delta \in \mathbb{R}^+ \) without altering the truthfulness of the statement below?

\[
\forall x \in \mathbb{R}, \forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+, |x - 0.6| < \delta \Rightarrow |x^2 - 0.36| < \varepsilon
\]

(you can!). How about:

\[
\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+, \forall x \in \mathbb{R}, |x - 0.6| < \delta \Rightarrow |x^2 - 0.36| < \varepsilon
\]

This latter is often written in a different form:

\[
\lim_{x \to 0.6} x^2 = 0.36
\]

First specify how close to 0.36 \( x^2 \) has to be (\( \varepsilon \)), then I can choose how close to 0.6 \( x \) must be (\( \delta \)). If I choose \( \delta \) first, can it work for all \( \varepsilon \)?
graphically...
are we close to infinity yet?

What is meant by phrases such as “as $x$ approaches (gets close to) infinity, $x^2$ increases without bound (sometimes ’becomes infinite’)”? Or even

$$\lim_{x \to \infty} x^2 = \infty$$

Look at the graph of $x^2$. Do either $x$ or $x^2$ ever reach infinity?

How about:

$$\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+, \forall x \in \mathbb{R}, x > \delta \Rightarrow x^2 > \varepsilon$$

Getting “close” to infinity means getting far from (and greater than) zero. Once you have a specification for how far from zero $x^2$ must be ($\varepsilon$), you can come up with how far from zero $x$ must be ($\delta$). Can you choose a $\delta$ in advance that works for all $\varepsilon$?
graph “approaching infinity”
asymptotic
double quantifiers

There are (at least) three ways to claim that a certain subset of the cartesian product $\mathbb{N} \times \mathbb{N}$, aka $\mathbb{N}^2$, is non-empty:

\[
\exists m \in \mathbb{N}, \exists n \in \mathbb{N}, m^2 = n
\]
\[
\exists (m, n) \in \mathbb{N}^2, m^2 = n
\]
\[
\exists n \in \mathbb{N}, \exists m \in \mathbb{N}, m^2 = n
\]

Whether we think of this as a statement about a subset of the cartesian product being empty, or a relation between non-empty subsets of $\mathbb{N}$, it is symmetrical.

There are (at least) three ways to claim that the entire cartesian product $\mathbb{N} \times \mathbb{N}$ has some property:

\[
\forall m \in \mathbb{N}, \forall n \in \mathbb{N}, mn \in \mathbb{N}
\]
\[
\forall (m, n) \in \mathbb{N}^2, mn \in \mathbb{N}
\]
\[
\forall n \in \mathbb{N}, \forall m \in \mathbb{N}, mn \in \mathbb{N}
\]

Again, the order in which we consider elements of an ordered pair doesn’t change the logic.
\[
\neg \left[ \left( (p \Rightarrow q) \land (q \Rightarrow r) \right) \land \neg p \land \neg r \right] \\
\equiv \left[ (\neg p \lor q) \land (\neg q \lor \neg r) \right] \land p \land \neg r \\
\equiv \left[ (\neg p \lor q) \land \neg q \lor \neg r \right] \land p \land \neg r \\
\equiv \left[ (\neg p \lor q) \land \neg q \lor \neg r \right] \land (p \land \neg r) \\
\equiv \left[ (\neg p \lor q) \land \neg q \lor \neg r \right] \land F \\
\equiv \left[ \neg p \land \neg q \land \neg r \right] \lor \neg p \land \neg r \lor q \land \neg r \\
\equiv F
\]
Notes

Prove:

\[ \frac{\rho \Rightarrow (Q \Rightarrow R)}{\rho \Rightarrow (Q \Rightarrow R)} \equiv Q \Rightarrow (\rho \Rightarrow R) \]

you may use truth tables or De Morgan's

& other identities ...

\[ \neg \rho \lor (\neg Q \lor R) \equiv \text{identity} \]

\[ \equiv (\neg \rho \lor \neg Q) \lor R \equiv \text{associative} \]

\[ \equiv (\neg Q \lor \neg \rho) \lor R \equiv \text{commutative} \]

\[ \equiv \neg Q \lor (\neg \rho \lor R) \equiv \text{associative} \]

\[ \equiv Q \Rightarrow (\rho \Rightarrow R) \equiv \lor \text{identity} \]

done