QUESTION 1.  [10 marks]

Consider the definition of $U(n)$:

$$U(n) \iff \exists i \in \mathbb{N}, n = 6i + 3.$$ 

Use the definition to prove $\forall n \in \mathbb{N}, U(n) \Rightarrow U(n^2)$.

SAMPLE SOLUTION:

Assume $n$ is a generic natural number. \# in order to introduce $\forall n$

Assume $U(n)$. \# antecedent

Then $\exists i' \in \mathbb{N}, n = 6i' + 3$. \# by definition of $U(n)$

Pick $i \in \mathbb{N}, n = 6i + 3$. \# since it exists

Then $n^2 = (6i + 3)^2 = 6(6i^2 + 6i + 1) + 3$ \# substitute $i$ and expand

Then $\exists j \in \mathbb{N}, n^2 = 6j + 3$. \# $j = 6i^2 + 6i + 1 \in \mathbb{N}$, since $6, i, 1 \in \mathbb{N}$ and $\mathbb{N}$ closed under $+$, $\times$.

Then $U(n^2)$. \# satisfies definition

Then $U(n) \Rightarrow U(n^2)$. \# introduced $\Rightarrow$

Then $\forall n \in \mathbb{N}, U(n) \Rightarrow U(n^2)$. \# introduced $\forall n$

QUESTION 2.  [10 marks]

Consider the definition of the sequence $a_n$:

$$a_n = \begin{cases} 
0, & \text{if } \exists i \in \mathbb{N}, n = 3i \\
1, & \text{if } \exists i \in \mathbb{N}, n = 3i + 1 \\
2, & \text{if } \exists i \in \mathbb{N}, n = 3i + 2 
\end{cases}$$

Use the definition of $a_n$ to DISPROVE $\exists m \in \mathbb{N}, \forall n \in \mathbb{N}, a_m \geq a_n$.

SAMPLE SOLUTION: To disprove the statement, I disprove its negation

$$\forall m \in \mathbb{N}, \exists n \in \mathbb{N}, a_m \leq a_n$$

Assume $m \in \mathbb{N}$. \# in order to introduce $\forall m$

Pick $n = m$. Then $n \in \mathbb{N}$. \# since $m \in \mathbb{N}$

Then $a_n = a_m$. \# same indices

Then $a_n \leq a_m$. \# $\leq$ includes $=$

Then $\exists n \in \mathbb{N}, a_n \leq a_m$. \# previous line

Then $\forall m \in \mathbb{N}, \exists n \in \mathbb{N}, a_n \leq a_m$. \# introduced $\forall m$

REMARK: I inadvertently made this question slightly easier than I intended. Choose $n = m + 3$ or $n = 2$ would also work fine.
QUESTION 3. [10 marks]

Consider the definition of \( [x] \):

\[
y = [x] \iff y \in \mathbb{Z} \land y \leq x \land (\forall z \in \mathbb{Z}, z \leq x \Rightarrow z \leq y)
\]

Use the definition to prove \( \forall x \in \mathbb{R}^+, [3x] \geq [x] \).

SAMPLE SOLUTION:

Assume \( x \in \mathbb{R}^+ \). # in order to introduce \( \forall x \).

Then \( x \leq 3x \). # multiply 1 \( \leq 3 \) by positive \( x \)
Then \( [x] \leq x \). # definition of \( [x] \)
Then \( [x] \leq 3x \). # transitivity of \( \leq \)
Then \( [x] \in \mathbb{Z} \). # definition of \( [x] \).
Then \( [x] \leq [3x] \).

# Since any integer no greater than \( 3x \) is no greater than \( [3x] \), by the definition of \( [3x] \).

Then \( \forall x \in \mathbb{R}^+, [x] \leq [3x] \). # introduced \( \forall x \).