CSC 165
asymptotics
Course Notes chapter 5
Danny Heap
heap@cs.toronto.edu

http://www.cdf.toronto.edu/~heap/165/W10/

E3 - now posted, due Wednesday 17th
A3 - " , due " 24th
T3 - coming...
sort strategies

Which algorithm do you use to sort a 5-card euchre hand?

- insertion sort
- selection sort
- some other sort?

If you use one of the first two, the number of “steps” you execute will more than quadruple if you graduate from euchre to a 13-card bridge hand.

If you were enough of a virtuoso to use mergesort or quicksort on your cards, the change from euchre to bridge would roughly double your work.

We are most interested in how quickly running-time grows with the size of the problem, since these quickly swamp constant-factor differences between algorithms that are of the same “order.”
different, but the same

Suppose you could count the “steps” required by an algorithm in some sort of platform-independent way. You would find that the steps required for insertion sort and selection sort on lists of size $n$ were no more than some quadratic functions of $n$.

To a computer scientist, even though they may vary by substantial constant factors, all quadratic functions are the “same” — they are in $O(n^2)$.

\[ g(n) = n^2 \quad f(n) = 3n^2 + 50 \quad h(n) = 15n^2 + n \]
big-Oh of $n^2$

We know, or have heard, that all quadratic functions grow at “roughly” the same speed. Here’s how we make “roughly” explicit.

\[ \mathcal{O}(n^2) = \{ f : \mathbb{N} \rightarrow \mathbb{R}^\geq \mid \exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow f(n) \leq cn^2 \} \]

Those are a lot of symbols to process. They say that $\mathcal{O}(n^2)$ is a set of functions that take natural numbers as input and produce non-negative real numbers as output. An additional property of these functions is that for each of them you can find a multiplier $c$, and a breakpoint $B$, so that if you go far enough to the right (beyond $B$) the function is bounded above by $cn^2$.

In terms of limits, this says that as $n$ approaches infinity, $f(n)$ is no bigger than $cn^2$ (once you find the appropriate $c$).
\[ \exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, \ n \geq B \implies 3n^2 + 2n \leq cn^2 \]

Prove \( 3n^2 + 2n \in O(n^2) \)

Pick \( c = \frac{4}{f} \). Then \( c \in \mathbb{R}^+ \)

Pick \( B = \frac{\sqrt{2}}{1} \). Then \( B \in \mathbb{N} \).

Assume \( n \) is a natural number \# a typical one...

Assume \( n \geq B \)

Then \( 3n^2 + 2n \leq 3n^2 + n^2 \)

\[ \begin{align*}
3n^2 & + 2n \\
\leq & 3n^2 + n^2 \\
= & 4n^2 \\
= & cn^2
\end{align*} \]

\# Since \( c = 4 \)

Then \( n \geq B \implies 3n^2 + 2n \leq cn^2 \)

Then \( \forall n \in \mathbb{N}, \ n \geq B \implies 3n^2 + 2n \leq cn^2 \)

Then \( \exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \ n \geq B \implies 3n^2 + 2n \leq cn^2 \)

\# intro \( \implies \)

\# intro \( \forall n \)

\# intro \( \exists 2x \)

Conclude \( 3n^2 + 2n \in O(n^2) \) \# satisfies defn.
scratch
Was the first big-Oh exercise a special case? What happens to the argument if you add a constant:

Prove $3n^2 + 2n + 5 \in \mathcal{O}(n^2)$

See what needs to be modified in the proof to accommodate the constant 5.
in general, $O(g)$:

$$O(g) = \{ f : N \to \mathbb{R}^+ \mid \exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow f(n) \leq cg(n) \}$$

Prove:

$$7n^6 - 5n^4 + 2n^3 \in O(6n^8 - 4n^5 + n^2)$$

Pick $c = \frac{9}{2}$. Then $c \in \mathbb{R}^+$.

Pick $B = \frac{1}{2}$. Then $B \in \mathbb{N}$.

Assume $n \in \mathbb{N}$ and $n \geq B$ # generic $n$ and antecedent.

Then $7n^6 - 5n^4 + 2n^3 \leq 7n^6 + 2n^3 \leq 7n^6 + 2n^6 \leq 9n^6$. # subtract non-pos quantity

# $2n^6 \geq 2n^3$, $\forall n \in \mathbb{N}$

# $n^3 \geq 1$, mult by $2n^3$

$$= \frac{9}{2}n^6 \Rightarrow c = \frac{9}{2}$$

$$\leq \frac{9}{2}n^6 \Rightarrow c = \frac{9}{2}$$

$$\leq \frac{9}{2} \left( 6n^8 - 4n^5 + 1 \right) \Rightarrow n^2 \geq 0$$

Conclusions here

slide 8
scratch
how to prove $n^3 \not\in O(3n^2)$?

Assume $c \in \mathbb{R}^+$ and $B \in \mathbb{N}$ # generic real, natural.

Pick $n = \lceil 3c \rceil + 1 + B$. Then $n \geq B \land n \in \mathbb{N}$

Then $n^3 = nn^2 > c3n^2$ # since $n > 3c$

Then $\forall c \in \mathbb{R}^+$, $\forall B \in \mathbb{N}$, $\exists n \in \mathbb{N}$, $n \geq B \land n^3 > c3n^2$

# introduced $\forall 2x, \exists one, n \geq B, \land n^3 > c3n^2$.

Conclude $n^3 \not\in O(3n^2)$ # violates definition.
scratch
non-polynomials

Big-oh statements about polynomials are pretty easy to prove: \( f \in \mathcal{O}(g) \) exactly when the highest-degree term of \( g \) is no smaller than the highest-degree term of \( f \).

What about functions such as \( \log(n) \) or \( 3^n \)? Logarithmic functions are in big-Oh of ANY polynomial, whereas exponential functions (with a base bigger than 1) are not in big-Oh of any polynomial. How do you prove such things?
the long way

Without the techniques of calculus, you could prove that $2^n \not\in \mathcal{O}(n^2)$. The key idea is that you’d have to show that for ANY given $c$, you could find an $n$ so that $2^n > cn^2$.

To make this work nicely, it would be nice to have a piece of $n$ to overwhelm any multiplier $c$ that could be thrown at us in the form of $cn^2$.

For example, it would be convenient if $2^n$ were greater than $n^3 = nn^2$. This certainly isn’t true for all $n$, but is it true “eventually”?

\[
\frac{3^k}{8} = 2^n \quad ? \quad n^3
\]

Exercise: find natural number $k$ so that you can prove that for all natural numbers greater than $k$, $2^n > n^3$. 

slide 13
Proof:

Assume \( c \in \mathbb{R}^+ \) and \( b \in \mathbb{N} \). \( # \) generic \( + \) real, natural.

Pick \( n = \left[ \frac{c}{1-c} + 9 + B \right] \). Then \( n \in \mathbb{N} \land n \geq B \).

Then \( 2^n > n^3 \) if \( n > 9 \), using

Then \( \forall c \in \mathbb{R}^+, \forall B \in \mathbb{N}, \exists n \in \mathbb{N}, n \geq B \land 2^n > ch^2 \).

Introduce \( \forall 2^x, \exists n \geq B \land 2^n > ch^2 \).

Conclude \( 2^n \notin \Theta(n^2) \).

\[
\lim_{n \to \infty} \frac{2^n}{n^2} = \lim_{n \to \infty} \frac{\ln 2 \cdot 2^n}{2n} = \lim_{n \to \infty} \frac{(\ln 2)^2 \cdot 2^n}{2} = \infty
\]
scratch
the short cut

Do you know anything about the ratio $\frac{2^n}{n^2}$, as $n$ gets very large? How do you evaluate:

$$\lim_{n \to \infty} \frac{2^n}{n^2} = \infty$$

If the limit evaluates to $\infty$, then that's the same as saying:

$$\forall c \in \mathbb{R}^+, \exists n' \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq n' \Rightarrow \frac{2^n}{n^2} > c$$

Once your enemy hands you a $c$, you can choose an $n'$ with the required property.
Prove $2^n \not\in n^2$.
(using l'Hôpital's rule and limits)

Assume $c \in \mathbb{R}^+$ and $B \in \mathbb{N}$. A generic + real, not numb.

Then $\exists n' \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq n' \Rightarrow \frac{2^n}{n^2} > c$ # limit

Pick $n' \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq n' \Rightarrow \frac{2^n}{n^2} > c$

Pick $n = n' + B + 1$. Then $n \in \mathbb{N} \land n \geq B$.

Then $\frac{2^n}{n^2} > c$

Then $2^n > cn^2$ \(\# n > 0\)
scratch