sort strategies

Which algorithm do you use to sort a 5-card euchre hand?

- insertion sort
- selection sort
- some other sort?

If you use one of the first two, the number of “steps” you execute will more than quadruple if you graduate from euchre to a 13-card bridge hand.

If you were enough of a virtuoso to use mergesort or quicksort on your cards, the change from euchre to bridge would roughly double your work.

We are most interested in how quickly running-time grows with the size of the problem, since these quickly swamp constant-factor differences between algorithms that are of the same “order.”
different, but the same

Suppose you could count the “steps” required by an algorithm in some sort of platform-independent way. You would find that the steps required for insertion sort and selection sort on lists of size \( n \) were no more than some quadratic functions of \( n \):

\[
g(n) = n^2 \quad f(n) = 3n^2 + 50 \quad h(n) = 15n^2 + n
\]

To a computer scientists, even though they may vary by substantial constant factors, all quadratic functions are the “same” — they are in \( \mathcal{O}(n^2) \).
big-Oh of $n^2$

We know, or have heard, that all quadratic functions grow at “roughly” the same speed. Here’s how we make “roughly” explicit.

$$\mathcal{O}(n^2) = \{ f : \mathbb{N} \mapsto \mathbb{R}^+ \mid \exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow f(n) \leq cn^2 \}$$

Those are a lot of symbols to process. They say that $\mathcal{O}(n^2)$ is a set of functions that take natural numbers as input and produce non-negative real numbers as output. An additional property of these functions is that for each of them you can find a multiplier $c$, and a breakpoint $B$, so that if you go far enough to the right (beyond $B$) the function is bounded above by $cn^2$.

In terms of limits, this says that as $n$ approaches infinity, $f(n)$ is no bigger than $cn^2$ (once you find the appropriate $c$).
Prove $3n^2 + 2n \in \mathcal{O}(n^2)$
scratch
Special case?

Was the first big-Oh exercise a special case? What happens to the argument if you add a constant:

\[ 3n^2 + 2n + 5 \in O(n^2) \]

See what needs to be modified in the proof to accommodate the constant 5.
in general, $\mathcal{O}(g)$:

$$\mathcal{O}(g) = \{ f : \mathbb{N} \mapsto \mathbb{R}^\geq \mid \exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow f(n) \leq cg(n) \}$$

Prove:

$$7n^6 - 5n^4 + 2n^3 \in \mathcal{O}(6n^8 - 4n^5 + n^2)$$
scratch
how to prove $n^3 \not\in \mathcal{O}(3n^2)$?
scratch
non-polynomials

Big-oh statements about polynomials are pretty easy to prove: $f \in O(g)$ exactly when the highest-degree term of $g$ is no smaller than the highest-degree term of $f$.

What about functions such as $\log(n)$ or $3^n$? Logarithmic functions are in big-Oh of ANY polynomial, whereas exponential functions (with a base bigger than 1) are not in big-Oh of any polynomial. How do you prove such things?
the long way

Without the techniques of calculus, you could prove that \(2^n \not\in \mathcal{O}(n^2)\). The key idea is that you’d have to show that for ANY given \(c\), you could find an \(n\) so that \(2^n > cn^2\).

To make this work nicely, it would be nice to have a piece of \(n\) to overwhelm any multiplier \(c\) that could be thrown at us in the form of \(cn^2\).

For example, it would be convenient if \(2^n\) were greater than \(n^3 = nn^2\). This certainly isn’t true for all \(n\), but is it true “eventually”?

Exercise: find natural number \(k\) so that you can prove that for all natural numbers greater than \(k\), \(2^n > n^3\).
Prove $2^n \not\in n^2$. 
scratch
the short cut

Do you know anything about the ratio $2^n/n^2$, as $n$ gets very large? How do you evaluate:

$$\lim_{n \to \infty} \frac{2^n}{n^2}$$

If the limit evaluates to $\infty$, then that’s the same as saying:

$$\forall c \in \mathbb{R}^+, \exists n' \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq n' \Rightarrow \frac{2^n}{n^2} > c$$

Once your enemy hands you a $c$, you can choose an $n'$ with the required property.
Prove $2^n \not\in n^2$.
(using l’Hôpital’s rule and limits)
scratch