CSC 165
more proof
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http://www.cdf.toronto.edu/~heap/165/W10/
proving existence

To prove the a set is non-empty, it’s enough to exhibit one element. How do you prove:

$$\exists x \in \mathbb{R}, x^3 + 3x^2 - 4x = 12$$
prove a claim about a sequence

Define sequence $a_n$ by:

$$\forall n \in \mathbb{N} \quad a_n = n^2$$

Now prove:

$$\exists i \in \mathbb{N}, \forall j \in \mathbb{N}, a_j \leq i \Rightarrow j < i$$
infinitely many primes

Define the prime natural numbers as $P = \{p \in \mathbb{N} \mid p$ has exactly two distinct divisors in $\mathbb{N}\}$. How do you prove:

$$S : \forall n \in \mathbb{N}, |P| > n$$

It would be nice to have some result $R$ that leads to $S$. If you could show $R \Rightarrow S$, and that $R$ is true, then you’d be done. But, out of many elementary results, how do you choose an $R$? Contradiction will often lead you there.
scratch
non-boolean functions

Take care when expressing a proof about a function that returns a non-boolean value, such as a number:

\[ [x] \text{ is the largest integer } \leq x. \]

Now prove the following statement (notice that we quantify over \( x \in \mathbb{R} \), not \( [x] \in \mathbb{R} \)):

\[ \forall x \in \mathbb{R}, [x] < x + 1 \]
using more of the definition

You may have been disappointed that the last proof used only part of the definition of floor. Here's a symbolic re-writing of the definition of floor:

\[ \forall x \in \mathbb{R} \quad y = |x| \iff y \in \mathbb{Z} \land y \leq x \land (\forall z \in \mathbb{Z}, z \leq x \Rightarrow z \leq y) \]

The full version of the definition should prove useful to prove:

\[ \forall x \in \mathbb{R}, |x| > x - 1 \]
scratch
prove something is false

Define a sequence:

$$\forall n \in \mathbb{N} \quad a_n = \lfloor n/2 \rfloor$$

(of course, if you treat “/” as integer division, there’s no need to take the floor. Now consider the claim:

$$\exists i \in \mathbb{N}, \forall j \in \mathbb{N}, j > i \Rightarrow a_j = a_i$$

The claim is false. Disprove it.
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Sometimes your argument has to split to take into account possible properties of your generic element:

$$\forall n \in \mathbb{N}, n^2 + n \text{ is even}$$

A natural approach is to factor $n^2 + n$ as $n(n + 1)$, and then consider the case where $n$ is odd, then the case where $n$ is even.
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