CSC 165
mixed quantifiers
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\[ P \text{ if-then } Q \]

\[ \text{unless} \]

\[ [0, -3, 2] \]

\[ [4, 1, 4] \]

\[ P \text{ is nec. for } Q \]

\[ Q \implies P \]

\[ P \text{ is suf. for } Q \]

\[ \forall p \implies Q \]
The result of the following truth table is useful enough to bear restating:

<table>
<thead>
<tr>
<th>$P$</th>
<th>$Q$</th>
<th>$P \Rightarrow Q$</th>
<th>$\neg P \lor Q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
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</tbody>
</table>

The result: 

- Every combination of $\land, \lor, \rightarrow$ can be expressed using $\land, \lor, \neg$ (or NAND).
convincing yourself that the identities above are true using venn diagrams, truth tables, or expressing them in words. Some are analogous to arithmetic properties of numbers. Some are truly novel.
De Morgan’s law(s)

Really just one law, but you switch the roles of $\land$ and $\lor$:

$$\neg(P \lor Q) \Leftrightarrow \neg P \land \neg Q$$

$$\neg(P \land Q) \Leftrightarrow \neg P \lor \neg Q$$

Again, you should draw venn diagrams and fill in truth tables to convince yourself this is true. Using associativity and commutativity, you can extend these laws to conjunctions and disjunctions of more than two expressions.
Translate bi-implication into the conjunction of two disjunctions:

\[(P \Rightarrow Q) \land (Q \Rightarrow P)\]

Now change your expression for bi-implication into the disjunction of two conjunctions (use some of the equivalences from a few slides ago):

\[\neg[(\neg P \lor Q) \land (\neg Q \lor P)] \lor [\neg(\neg P \land Q) \land (P \land \neg Q)]\]

What's the negation of bi-implication? How would you explain it in English?

\[\neg \left[ (\neg P \land \neg Q) \lor (P \land Q) \right] \]
\[ \neg [ (\neg p \wedge \neg q) \vee (p \wedge q) ] \iff \neg (\neg p \wedge q) \iff \neg (p \vee q) \iff \neg (p \wedge \neg q) \iff \neg (p \wedge q) \iff p \wedge \neg q \iff \neg (p \iff q) \]

De Morgan’s

\[ (p \vee q) \wedge (\neg p \vee \neg q) \iff \neg (p \wedge q) \iff \neg (p \vee q) \iff \neg (p \wedge \neg q) \iff \neg (p \wedge q) \iff p \wedge \neg q \iff \neg (p \iff q) \]

Distributivity

\[ [(p \vee q) \wedge \neg p] \vee [(p \vee q) \wedge \neg q] \iff \neg (p \wedge q) \iff \neg (p \vee q) \iff \neg (p \wedge \neg q) \iff \neg (p \wedge q) \iff p \wedge \neg q \iff \neg (p \iff q) \]

Distributivity

\[ [(p \vee q) \wedge (q \vee \neg p)] \vee [(p \vee q) \wedge (q \vee \neg q)] \iff \neg (p \wedge q) \iff \neg (p \vee q) \iff \neg (p \wedge \neg q) \iff \neg (p \wedge q) \iff p \wedge \neg q \iff \neg (p \iff q) \]

\[ (q \vee \neg p) \vee (p \vee \neg q) \iff p \wedge \neg q \iff p \wedge \neg q \iff \neg (p \iff q) \]
What does the following statement mean, when you interpret it as a Venn diagram?

\[ \forall x \in X, (P(x) \Rightarrow Q(x)) \land (Q(x) \Rightarrow R(x)) \]

\[ p \subseteq q \land q \subseteq r \]

\[ \Rightarrow p \subseteq r \]

For another insight, negate the following statement, and simplify it by transforming implications into disjunctions:

\[ ((P \Rightarrow Q) \land (Q \Rightarrow R)) \Rightarrow (P \Rightarrow R) \]

\[ \neg \left[ ((P \Rightarrow Q) \land (Q \Rightarrow R)) \Rightarrow (P \Rightarrow R) \right] \]
\[ \neg [((p \Rightarrow q) \land (q \Rightarrow r)) \Rightarrow (p \Rightarrow r)] \iff \# \rightarrow \nabla \]

\[ (p \Rightarrow q) \land (q \Rightarrow r) \land p \land \neg r \iff \# \]

\[ ((p \lor q) \land (\neg q \lor r)) \land p \land \neg r \iff \# \text{ comm.} \]

\[ (\neg p \lor q) \land p \lor (\neg q \lor r) \land \neg r \iff \# \text{ identity} \]

\[ (\neg q \lor (p \lor r)) \land (\neg q \land \neg r) \lor (r \land r) \iff p \land q \land \neg q \land \neg r \]

\[ \Rightarrow \text{ False} \]
What's the difference between these two claims:

$$\forall x \in L_1, \exists y \in L_2, x + y = 5$$

$$\exists y \in L_2, \forall x \in L_1, x + y = 5$$

```python
def P(x, y): return x + y == 5
L1 = L2 = [1, 2, 3, 4]
def forallExists(P, L1, L2):
    return False not in [True in [P(x, y) for y in L2] for x in L1]
def existsForall(P, L1, L2):
    return True in [False not in [P(x, y) for x in L2] for y in L1]
```
dangerous switching

Can you switch $\forall \varepsilon \in \mathbb{R}^+ \text{ with } \exists \delta \in \mathbb{R}^+$ without altering the truthfulness of the statement?

This is often written in a different form:

$$\lim_{x \to 0.6} x^2 = 0.36$$

First specify how close to 0.36 $x^2$ has to be ($\varepsilon$), then I can choose how close to 0.6 $x$ must be ($\delta$). If I choose $\delta$ first, can it work for all $\varepsilon$?
graphically

\[ x^2 = 0.36 \]
\[ |x - 0.6| < 0.6 \]

\[ x^2 : (0.6 - \delta, 0.6 + \delta) \]

\[ (0.36 - \varepsilon, 0.36 + \varepsilon) \]

\( x^*_\delta \) neighbourhood

\( \varepsilon \)-neighbourhood

slide 11
are we close to infinity yet?

What is meant by phrases such as “as \( x \) approaches (gets close to) infinity, \( x^2 \) increases without bound (sometimes ‘becomes infinite’)”? Or even

\[
\lim_{x \to \infty} x^2 = \infty
\]

Look at the graph of \( x^2 \). Do either \( x \) or \( x^2 \) ever reach infinity?

How about:

\[
\forall x \in \mathbb{R}, \forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+, x > \delta \implies x^2 > \varepsilon
\]

Getting “close” to infinity means getting far from (and greater than) zero. Once you have a specification for how far from zero \( x^2 \) must be (\( \varepsilon \)), you can come up with how far from zero \( x \) must be (\( \delta \)). Can you choose a \( \delta \) in advance that works for all \( \varepsilon \)?
graph “approaching infinity”

$x^2: (N, \infty) \rightarrow (N, \infty)$

$x^2$ maps a $\varepsilon$-neighbourhood of $0$ to a $\delta$-neighbourhood of $\infty$.
Asymptotic

$V \epsilon \mathbb{R}^+$

$V \epsilon \mathbb{R}^+$

$E, \exists \epsilon \mathbb{R}^+$

$3 > | \frac{1}{x} | \iff | \frac{1}{3} - 0 | < 3$
double quantifiers

There are (at least) three ways to claim that a certain subset of the cartesian product \( \mathbb{N} \times \mathbb{N} \), aka \( \mathbb{N}^2 \) is non-empty:

\[
\exists m \in \mathbb{N}, \exists n \in \mathbb{N}, m^2 = n \\
\exists (m, n) \in \mathbb{N}^2, m^2 = n \\
\exists n \in \mathbb{N}, \exists m \in \mathbb{N}, m^2 = n
\]

Whether we think of this as a statement about a subset of the cartesian product being empty, or a relation between non-empty subsets of \( \mathbb{N} \), it is symmetrical.

There are (at least) three ways to claim that the entire cartesian product \( \mathbb{N} \times \mathbb{N} \) has some property:

\[
\begin{align*}
\forall m \in \mathbb{N}, \forall n \in \mathbb{N}, mn & \in \mathbb{N} \\
\forall (m, n) \in \mathbb{N}^2, mn & \in \mathbb{N} \\
\forall n \in \mathbb{N}, \forall m \in \mathbb{N}, mn & \in \mathbb{N}
\end{align*}
\]

Again, the order in which we consider elements of an ordered pair doesn’t change the logic.
A proof communicates why and how you believe something to be true. You'll need to master two things:

1. Why do you believe the thing is true? This step is messy, creative, but then increasingly precise to identify (and then strengthen) the weak parts of your belief.

2. Write up (express) why you believe the thing is true. Each step of your written proof should be justified enough to convince a skeptical peer. If you detect a gap in your reasoning, you may have to go back to step 1.

Although I present a great deal of symbolic notation, we will accept carefully-structured, precise English prose. The structure, however, is required, and is the main topic of Chapter 4.
finding a proof of universally-quantified $\Rightarrow$

To support a proof of a universally-quantified implication $\forall x \in X, P(x) \Rightarrow Q(x)$, you usually need to use some already-proven statements and axioms (defined, or assumed, to be true for $X$). You hope to find a chain

$$
\begin{align*}
C2.0 & \quad \forall x \in X, P(x) \Rightarrow R_1(x) \\
C2.1 & \quad \forall x \in X, R_1(x) \Rightarrow R_2(x) \\
& \quad \vdots \\
C2.n & \quad \forall x \in X, R_n(x) \Rightarrow Q(x)
\end{align*}
$$

Such a chain shows in $n$ steps that $P(x) \Rightarrow Q(x)$, by transitivity.
proof outline

A slightly more flexible format will be required in this course. Each link in the chain is justified by mentioning the supporting evidence in a comment beside it. We show the portions of the argument where an assumption is in effect by using indentation. Here's what a generic proof that \( \forall x \in X, P(x) \implies Q(x) \) might look like.

Assume \( x \in X \) \# x is generic, so what I prove about it applies to all of X

Assume \( P(x) \) \# Antecedent. Otherwise, \( \neg P(x) \) means we get the implication for free.

Then \( R_1(x) \) \# by previous result C2.0, \( \forall x \in X, P(x) \implies R_1(x) \)

Then \( R_2(x) \) \# by previous result C2.1, \( \forall x \in X, R_1(x) \implies R_2(x) \)

:  

Then \( Q(x) \) \# by previous result C2.n, \( \forall x \in X, R_n(x) \implies Q(x) \)

Then \( P(x) \implies Q(x) \) \# I assumed antecedent, got consequent (aka introduced \( \implies \))

Then \( \forall x \in X, P(x) \implies Q(x) \) \# reasoning about generic x works for all \( x \in X \).
tracking the wiley chain of results

The hard part is finding that chain of implications. Here are two models for your search (they are equivalent).

- **Bubble Search**: As a Venn diagram, you are searching for a chain of supersets from $P$ to $Q$. Work forwards (supersets of $P$) and backwards (subsets of $Q$). As soon as you find a set that's on both lists, you're done.

- **Tree Search**: As a directed graph, you are searching for a path from $P$ to $Q$. Work forwards (consequents of $P$) and backwards (antecedents of $Q$). As soon as you find a predicate that's on both lists, you're done.
scratch