vacuous truth

We’ve already separated implication from quantification, so we can make sense of

\[ P(x) \implies Q(x) \]

It’s true, except when \( P(x) \) is true and \( Q(x) \) is false. In particular, an implication is always true when the antecedent is false. For example, if your eyes wander to the consequent in

\[
\forall x \in \mathbb{R}, x^2 - 2x + 2 = 0 \implies (x > x + 5)
\]

\[ \exists x \in \mathbb{R} : x^2 - 2x + 2 = 0 \implies (x > x + 5) \]

\[ \text{let } x = 0 \]

\[ \mathbb{R} \implies \mathbb{C} \]

\[ 1x > 1x + 5 \]

…you could jump to the conclusion that the implication is false.

Vacuous truth works because there are no counterexamples. Another way of thinking about this is that the empty set is a subset of every other set.

All employees earning over 80 trillion dollars are female.
All employees earning over 80 trillion dollars are male.
All employees earning over 80 trillion dollars have mauve eyeballs and breathe ammonia.
Suppose Al quits. Now consider the statement:

Every male employee earns between 25,000 and 45,000.

Is the statement true? What about its converse?

<table>
<thead>
<tr>
<th>EMPLOYEE</th>
<th>GENDER</th>
<th>SALARY</th>
</tr>
</thead>
<tbody>
<tr>
<td>Betty</td>
<td>female</td>
<td>500</td>
</tr>
<tr>
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<td>40,000</td>
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<tr>
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<td>30,000</td>
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</tr>
<tr>
<td>Gwen</td>
<td>female</td>
<td>95,000</td>
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An employee is male if, and only if, that employee earns 25,000–45,000. This is a double implication, $P \Rightarrow Q$ and $Q \Rightarrow P$, or $P \Leftrightarrow Q$. Thought of as sets, they are equal (mutual subsets).
How do you feel about
\[ \forall x \in \mathbb{R}, x^2 - 2x + 2 = 0 \iff x > x + 5. \]

Break it into two implications:
\[ \forall x \in \mathbb{R}, x^2 - 2x + 2 = 0 \Rightarrow x > x + 5. \]
\[ \forall x \in \mathbb{R}, x > x + 5 \Rightarrow x^2 - 2x + 2 = 0. \]

The truth values are the same. English phrases:
P is necessary and sufficient for Q.
P is true exactly when Q is true.
P implies Q, and conversely.
\[ P \Rightarrow Q, \text{ and conversely.} \]
\[ Q \Rightarrow P. \]

\[ P \quad \text{False make } (P \Rightarrow Q) \text{ true.} \]

more equivalence

\[ P \iff Q \quad \text{and conversely.} \]
symbolic idiom

Some expressions for restricting domains are more common than others.

- “Every $D$ that is a $P$ is also a $Q$.” Usually $\forall x \in D, P(x) \Rightarrow Q(x)$. Less common $\forall x \in D \cap P, Q(x)$. What about $\forall x \in D, P(x) \land Q(x)$ ($\land$ means “and”)?

- “Some $D$ that is a $P$ is also a $Q$.” Usually $\exists x \in D, P(x) \land Q(x)$. Less common $\exists x \in D \cap P \cap Q$. What about $\exists x \in D, P(x) \Rightarrow Q(x)$?
conjunction

Combine two statements by claiming they are both true with logical “and”:

\[ A(x) \text{ and } B(x) \] (python keyword \texttt{AND} works like this)
\[ A(x) \land B(x) \] (\(\land\) is a symbol for “and”)

As sets: \( x \in A \cap B \)

Notice that a conjunction is \textbf{false} if either part is false. “The employee makes less than 100,000 and more than 60,000,” is true for Gwen, but false for Ellen.

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watch out for English...

Sometimes the English word "and" is used to smear some meaning over several components:

There is a pen and a telephone.

In the universe of objects, \( O \), with predicates \( P(x) \) (\( x \) is a pen) and \( T(x) \) (\( x \) is a telephone), you could try to translate this as \( \exists x \in O, P(x) \land T(x) \). What’s a better translation into symbols?

\[
(\exists x \in O, P(x)) \land (\exists x \in O, T(x))
\]

Occasionally English usage of AND will differ from logical usage even in mathematical material:

The solutions are \( x < 10 \) and \( x > 20 \)

The solutions are \( x < 20 \) and \( x > 10 \)

The first statements probably meant the union of the two sets, or the logical OR. The second meant the intersection, so the logical AND is appropriate.
Combine two statements by claiming that at least one of them is true using OR (\(\lor\) in symbols).

- \(A(x) \lor B(x)\) (the python keyword or works like this)
- \(A(x) \lor B(x)\) (in symbols)
- \(x \in A \cup B\) (as sets)

Notice the close connection between the symbols for conjunction and intersection, \(\land\), \(\cap\), and the symbols for disjunction and union, \(\lor\), \(\cup\). Coincidence? In any case, you may use it as a mnemonic.

“The employee is female or earns more than 35,000.”

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more silly English tricks

In logic we use or generously, or inclusively, to mean something like “and/or”. Sometimes we convey the INCLUSIVE OR by saying something like “A or B, or both.” Be aware that natural English sometimes uses or to mean “A or B, but not both” — something we’d call EXCLUSIVE OR in logic:

Either we play the game my way, or I’m taking my ball and going home.
Negate the statement “All employees earning over 110,000 are female.” Usually prepending the word “Not” will work, and in logic we use the corresponding symbol $\neg$:

$$
\neg(\forall e \in E, O(e) \Rightarrow F(e))
$$

A good exercise is to “work” the negation $\neg$ as far into the statement as possible. The statement is true exactly when its negation is false.

The original statement is universally quantified, so it says something about an absence of counterexamples. The negation of the original statement should claim something about the presence of counterexamples.
Negating implications is a common task. There are several equivalent ways of doing this, but some are more common than others. Try negating the following in such a way that the $\neg$ symbol applies to the “smallest possible” part of the expression:

$$\neg \left( \forall x \in X, P(x) \Rightarrow Q(x) \right)$$

$$\exists x \in X, \neg P(x) \lor \neg Q(x)$$

Now for symmetry, negate the following in such a way that the $\neg$ symbol applies to the “smallest possible” part of the expression:

$$\neg \left( \exists x \in X, P(x) \land Q(x) \right)$$

$$\iff \forall x \in X, \neg (P(x) \land Q(x))$$

$$\iff \forall x \in X, P(x) \Rightarrow \neg Q(x)$$

$$Q(x) \Rightarrow \neg P(x)$$
going negative on logic

Negated expressions have some standard transformations:

- $-\forall x \in X, \ldots \Leftrightarrow \exists x \in X, -\ldots$
- $-\exists x \in X, \ldots \Leftrightarrow \forall x \in X, -\ldots$
- $- (P(x) \Rightarrow Q(x)) \Leftrightarrow P(x) \land -Q(x)$
- $- (P(x) \land Q(x)) \Leftrightarrow P(x) \Rightarrow -Q(x)$ (has this become asymmetrical?)

Push the $-\$ symbol “as far in” to the following expression as possible:

$$- (\forall x \in X, \exists y \in Y, P(x) \Rightarrow Q(x))$$

$$\Leftrightarrow (\exists x \in X), \forall y \in Y, P(x, y) \land -Q(x, y)$$
In order to parse a logical expression we need to know which subexpressions to parse first. Although there’s often conventions (just as in your favourite programming language), such as evaluating $\land$ before $\Rightarrow$, when in doubt you should use parentheses:

$$P(x) \lor Q(x) \Rightarrow R(x) : \quad (P(x) \lor Q(x)) \Rightarrow R(x) \quad \text{versus} \quad P(x) \lor (Q(x) \Rightarrow R(x))$$

This becomes particularly important when you want to be explicit about the scope of universal quantifiers:

$$\left( \forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x < y \right) \Rightarrow \left( \forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x^2 < y \right)$$

Notice that the scope of the quantification is inside the relevant parentheses. There’s no reason that the $y$ in the antecedent would be the same as the $y$ in the consequent. It could be re-written:

$$\left( \forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x < y \right) \Rightarrow \left( \forall z \in \mathbb{R}, \exists w \in \mathbb{R}, z^2 < w \right)$$
As conjunctions, disjunctions, negations, and other combinations of predicates become more ornate, we need help to interpret them. To think about the truth value of up to three predicates, you can probably draw a Venn diagram. For example, draw the Venn diagram showing which regions could NOT have any elements, and still remain consistent with $P(x) \Rightarrow (Q(x) \Rightarrow R(x))$.

How would you draw the analogous diagram for predicates $P, Q, R,$ and $S$? Perhaps if your 3D rendering skills were pretty good you’d manage. However, to combine more predicates, you need a new tool.
The standard venn diagram for 3 sets has $2^3$ regions — one region for each possible combination of truth values for its component sets. We can get the same effect with a rectangular diagram, or table:

<table>
<thead>
<tr>
<th>P</th>
<th>Q</th>
<th>R</th>
<th>$(Q \Rightarrow R)$</th>
<th>$P \Rightarrow (Q \Rightarrow R)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
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</table>

As an exercise, compare this to the table for $(P \land Q) \Rightarrow R$. What do you conclude?
You may have been unsettled in the previous slides that there were no domains stated for $P$, $Q$, or $R$, no definitions for them, and nothing about what arguments (if any) these predicates take. The reason this was okay was that we considered all 8 possible truth values for $P$, $Q$, and $R$—all possible logical “worlds” that matter in their case.

An example to help think about this is to consider all possible domains $D$ that $P$ or $Q$ could be part of, and all possible meanings for predicates $P$ or $Q$. Consider this very general situation:

$$
\forall D \in \mathcal{D}, \forall P \in \mathcal{P}(D), \forall Q \in \mathcal{P}(D), \forall x \in D, (P(x) \implies Q(x)) \iff (\neg P(x) \lor Q(x))
$$

Although there are infinitely many domains in $\mathcal{D}$, and infinitely many meanings for predicates $P$ and $Q$, there are only four lines in the relevant truth table, and the statement is true in all four.
weaker and weirder

The situation on the previous slide was a tautology — the statement is true in every possible world.

\[
\exists D \in \mathcal{D}, \exists P \in \mathcal{P}(D), \exists Q \in \mathcal{P}(D), \exists x \in D, (P(x) \implies Q(x)) \iff (Q(x) \implies P(x))
\]

...it’s possible to concoct a world where the statement is true. We say it’s satisfiable.

What about a statement that can’t every be true no matter what world we devise:

\[
\forall D \in \mathcal{D}, \forall P \in \mathcal{P}(D), \forall x \in D, (P(x) \land \neg P(x))
\]

...This is unsatisfiable (aka a contradiction).
commutative, associative, distributive

Some laws of arithmetic have counterparts in logic and set operations:

<table>
<thead>
<tr>
<th>Identity</th>
<th>Identity</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P \land Q \iff Q \land P$</td>
<td>$P \lor Q \iff Q \lor P$</td>
</tr>
<tr>
<td>$P \land (Q \land R) \iff (P \land Q) \land R$</td>
<td>$P \lor (Q \lor R) \iff (P \lor Q) \lor R$</td>
</tr>
<tr>
<td>$P \land (Q \lor R) \iff (P \land Q) \lor (P \land R)$</td>
<td>$P \lor (Q \land R) \iff (P \lor Q) \land (P \lor R)$</td>
</tr>
<tr>
<td>$P \land (Q \lor \neg Q) \iff P \iff P \lor (Q \land \neg Q)$</td>
<td>$P \land P \iff P \iff P \lor P$</td>
</tr>
</tbody>
</table>

Convince yourself that the identities above are true using venn diagrams, truth tables, or expressing them in words. Some are analogous to arithmetic properties of numbers. Some are truly novel.
De Morgan’s law(s)

Really just one law, but you switch the rôles of $\land$ and $\lor$:

\[
\neg(P \lor Q) \iff \neg P \land \neg Q \\
\neg(P \land Q) \iff \neg P \lor \neg Q
\]

Again, you should draw venn diagrams and fill in truth tables to convince yourself this is true. Using associativity and commutativity, you can extend these laws to conjunctions and disjunctions of more than two expressions.