1. It seems clear that $2^n$ is eventually greater than $n^3$. Find some natural number $k$ for which the following is true:

$$\forall n \in \mathbb{N} - \{0, \ldots, k - 1\}, 2^n > n^3$$

Once you’ve found an appropriate value for $k$, substitute it into the above claim and use mathematical induction to prove it.

The choice $k = 10$ will work, as will any value greater than 10. That means I can rewrite the statement as:

$$\forall n \in \mathbb{N} - \{0, \ldots, 9\}, 2^n > n^3$$

I prove the claim using simple induction.

**Base case:** $2^{10} = 1024 > 1000 = 10^3$, so the claim is verified when $n = 10$.

**Induction step:** Assume that $n$ is an arbitrary natural number greater than 9. # to introduce $\forall$

Assume $P(n)$, in other words $2^n > n^3$. # assume antecedent, or IH

Then

$$2^{n+1} = 2 \times 2^n$$

$$> 2n^3$$ # by IH

$$= n^3 + nn^2$$ # rewrite

$$\geq n^3 + 10n^2$$ # since $n \geq 10$

$$= n^3 + 3n^2 + n^2 + 6n^2$$ # rewrite some more

$$\geq n^3 + 3n^2 + 10n + 60n$$

# multiply $n \geq 10$ by $n$ and $6n$, yields $n^2 \geq 10n$ and $6n^2 \geq 60n$

$$> n^3 + 3n^2 + 3n + 1$$

# multiply $10 > 3$ by $n$ yields $10n > 3n$, and $n \geq 10$ by 60 yields $60n \geq 600 > 1$

$$= (n + 1)^3$$ # algebra

Then $2^{n+1} > (n + 1)^3$. # by the chain of inequalities above

Then $2^n > n^3 \Rightarrow 2^{n+1} > (n + 1)^3$ # assumed antecedent, derived consequent

Then $\forall n \in \mathbb{N} - \{0, \ldots, 9\}, 2^n > n^3 \Rightarrow 2^{n+1} > (n + 1)^3$ # introduced $\forall$

Conclude $\forall n \in \mathbb{N} - \{0, \ldots, 9\}, 2^n > n^3$. # proved by induction
2. Given polynomials \( f(n) = 9n^3 - 2n + 9 \) and \( g(n) = 5n^3 - 3n^2 + 7 \), prove that \( f \in O(g) \) by showing

\[
\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow f(n) \leq cg(n)
\]

**Sample solution:** The main idea is to work each end of the inequality until you are comparing two monomials (single-term polynomials).

**Proof:**

Pick \( c = 9 \). Then \( c \in \mathbb{R}^+ \).

Pick \( B = 1 \). Then \( B \in \mathbb{N} \).

Assume \( n \in \mathbb{N} \). # generic natural number

Assume \( n \geq B \). # antecedent

Then

\[
f(n) = 9n^3 - 2n + 9 \\
\leq 9n^3 + 9 \quad \# 0 \leq n, \text{ so } 0 \leq 2n, \text{ add } f(n) \text{ to both sides} \\
\leq 9n^3 + 9n^3 = 18n^3 \quad \# 9n^3 - 9 \geq 0, \text{ since } n \geq 1, \text{ so } n^3 \geq 1. \\
= 2cn^3 \quad \# \text{ since } c = 9 \\
\leq c(5n^3 - 3n^2) \quad \# \text{ since } n \geq 1 \Rightarrow 3n^3 \geq 3n^2, \text{ so } 5n^3 - 3n^2 \geq 5n^3 - 3n^3 \\
< c(5n^3 - 3n^2 + 7) = cg(n) \quad \# \text{ add } 5n^3 - 3n^2 \text{ to both sides of } 0 < 7
\]

Then \( n \geq B \Rightarrow f(n) \leq cg(n) \) # assume antecedent, derived consequent

Then \( \forall n \in \mathbb{N}, n \geq B \Rightarrow f(n) \leq cg(n) \) # introduced \( \forall \)

Then \( \exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow f(n) \leq cg(n) \) # introduced \( \exists \) twice

Conclude \( f \in O(g) \) # satisfies the definition.
3. Given polynomials \( f(n) = 2n + 3 \) and \( g(n) = 10\sqrt{n} \), prove that \( f \notin O(g) \) by showing

\[
\neg \left( \exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow f(n) \leq cg(n) \right)
\]

**Sample solution:** The negation I have to prove reads:

\[
\forall c \in \mathbb{R}^+, \forall B \in \mathbb{N}, \exists n \in \mathbb{N}, n \geq B \land f(n) > cg(n)
\]

The key idea is that \( 2n \) has two factors of \( \sqrt{n} \), so that I can always choose an \( n \) big enough to overshadow \( 10c\sqrt{n} \).

**Proof:** Assume \( c \in \mathbb{R}^+ \) and \( B \in \mathbb{N} \). # generic positive real and natural number.

Pick \( n = [25c^2] + B \). Then \( n \in \mathbb{N} \) and \( n \geq B \). # natural number no smaller than \( B \).

Then

\[
f(n) = 2n + 3
\]

\[
> 2n \quad \# \text{ add } 2n \text{ to both sides of } 3 > 0
\]

\[
= 2\sqrt{n}\sqrt{n} \quad \# \text{ rewrite}
\]

\[
\geq 10c\sqrt{n} = cg(n)
\]

\[
\# \quad n \geq 25c^2 \Rightarrow n + 5c\sqrt{n} \geq 25c^2 + 5c\sqrt{n} \Rightarrow \sqrt{n}(\sqrt{n} + 5c) \geq 5c(5c + \sqrt{n})
\]

\[
\# \quad \Rightarrow \sqrt{n} \geq 5c \Rightarrow 2\sqrt{n} \geq 10c \ldots \text{ but I'd accept } "\sqrt{n} \text{ is monotonic}"
\]

Then \( \exists n \in \mathbb{N}, n \geq B \land f(n) > cg(n) \) # introduced \( \exists \)

Then \( \forall c \in \mathbb{R}^+, \forall B \in \mathbb{N}, \exists n \in \mathbb{N}, n \geq B \land f(n) > cg(n) \) # introduced \( \exists \) twice

Conclude \( f \notin O(g) \). # it violates the definition.