You must make the choice between proving or disproving the claims below. If you are unsure of which choice to make, a good tactic is to write out the outline for a proof, and the outline for a disproof, and try to see which it is possible to complete.

1. Consider the following three predicates of the natural numbers:

   \[ \forall n \in \mathbb{N} \quad U(n) \iff \exists i \in \mathbb{N}, n = 5i + 2 \]
   \[ \forall n \in \mathbb{N} \quad V(n) \iff \exists j \in \mathbb{N}, n = 5j + 3 \]
   \[ \forall n \in \mathbb{N} \quad W(n) \iff \exists k \in \mathbb{N}, n = 5k + 4 \]

Use the proof structure from this course to prove or disprove each of the following claims:

(i) \( \forall n \in \mathbb{N}, \forall m \in \mathbb{N}, (V(m) \land W(n)) \Rightarrow U(mn) \).

**Sample solution:** The claim is true. Here's the proof:

Assume \( m \) and \( n \) are generic natural numbers.

Assume \( V(m) \land W(n) \). # assume the antecedent

Then \( \exists k' \in \mathbb{N}, m = 5k' + 3 \). # definition of \( V(m) \)

Pick \( k \in \mathbb{N}, m = 5k + 3 \). # since it exists

Then \( \exists j' \in \mathbb{N}, n = 5j' + 4 \). # definition of \( W(n) \)

Pick \( j \in \mathbb{N}, n = 5j + 4 \). # since it exists

Then \( mn = (5k + 3)(5j + 4) = 5(5jk + 4k + 3j + 2) + 2 \). # substitute and expand

Then \( \exists i \in \mathbb{N}, mn = 5i + 2 \).

# \( i = 5jk + 4k + 3j + 2 \in \mathbb{N} \), since \( 5, j, k, 4, 3, 2 \in \mathbb{N} \) and \( \mathbb{N} \) closed under +, \( \times \).

Then \( U(mn) \). # definition of \( U(mn) \).

Then \( (V(m) \land W(n)) \Rightarrow U(mn) \). # assumed antecedent, derived consequent

Conclude \( \forall n \in \mathbb{N}, \forall m \in \mathbb{N}, (V(m) \land W(n)) \Rightarrow U(mn) \). # introduced \( \forall \)
(ii) \( \forall m \in N, \forall n \in N, (U(m) \land W(n)) \Rightarrow V(mn) \).

**Sample solution:** The claim is true. Here’s the proof:

Assume \( m \) and \( n \) are generic natural numbers. \# in order to introduce \( \lor \)

Assume \( U(m) \land W(n) \). \# assume the antecedent

Then \( \exists i \in N, m = 5i + 2 \). \# definition of \( U(m) \)

Pick \( i \in N, m = 5i + 2 \). \# since it exists

Then \( \exists j \in N, n = 5j + 4 \). \# definition of \( W(n) \)

Pick \( j \in N, n = 5j + 4 \). \# since it exists

Then \( mn = (5i + 2)(5j + 4) = 5(5ij + 4i + 2j + 1) + 3 \). \# substitute, expand, simplify

Then \( \exists k \in N, mn = 5k + 3 \). \# introduced \( \Rightarrow \)

\( k = 5ij + 4i + 2j + 1 \in N \), since \( 5, i, j, 4, 2 \in N \), and \( N \) closed under \( +, \times \)

Then \( V(mn) \). \# definition of \( V(mn) \)

Then \( (U(m) \land W(n)) \Rightarrow V(mn) \)

Conclude \( \forall m \in N, \forall n \in N, (U(m) \land W(n)) \Rightarrow V(mn) \). \# introduced \( \lor \) twice

(iii) \( \forall m \in N, \forall n \in N, (U(m) \land V(n)) \Rightarrow W(mn) \).

**Sample solution:** The claim is false. The negation of the claim is:

\[ \exists m \in N, \exists n \in N, U(m) \land V(n) \land \neg W(mn) \]

... and that’s what I prove below.

Pick \( m = 2 \). Then \( m \) is a natural number. \# well-known member

Then \( \exists i \in N, m = 5i + 2 \). \# \( i = 0 \in N \)

Then \( U(m) \). \# definition of \( U(m) \)

Pick \( n = 3 \). Then \( n \) is a natural number. \# well-known member

Then \( \exists j \in N, n = 5j + 3 \). \# \( j = 0 \in N \)

Then \( V(n) \). \# definition of \( V(n) \)

Then \( mn = 6 = 5(1) + 1 \). \# substitute, expand, simplify

Then \( \neg \exists (q, r) \in N^2, 6 = 5q + r \land r \neq 1 \)

\# uniqueness of \( (q, r) \) in remainder theorem (see Mathematical Prerequisites)

Then \( \neg W(mn) \). \# \( \neg \exists q \in N, mn = 6 = 5q + 4 \), by previous line

Conclude \( \exists m \in N, \exists n \in N, U(m) \land V(n) \land \neg W(mn) \). \# conjunction of results introduced.

2. **Prove or disprove** the following claims, using the proof structure from this course. It may be useful to note that \( x^2 - y^2 = (x - y)(x + y) \)

(i) \( \forall x \in \mathbb{R}, \forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+, \forall y \in \mathbb{R}, |x - y| < \delta \Rightarrow |x^2 - y^2| < \varepsilon \).

**Sample solution:** The claim is true. Here’s the proof.

Assume \( x \in \mathbb{R} \) and \( \varepsilon \in \mathbb{R}^+ \). \# generic real number, generic positive real number

Pick \( \delta = \min(1, \varepsilon/(2(|2x| + 1))) \). Then \( \delta \in \mathbb{R}^+ \)

\# \( 1 \in \mathbb{R}^+ \) and \( \varepsilon/(2(|2x| + 1)) \in \mathbb{R}^+ \)

\# since \( \varepsilon, |2x| + 1, 2 \in \mathbb{R}^+ \) and \( \mathbb{R}^+ \) closed under \( +, \times, / \)

Assume \( y \in \mathbb{R} \). \# generic

Assume \( |x - y| < \delta \). \# assume antecedent


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Then
\[
|x^2 - y^2| = |(x - y)(x + y)| \quad \text{# factoring}
\]
\[
= |x - y| \times |x + y| \quad \text{# factor absolute values}
\]
\[
< \delta|x + y| \quad \text{# by antecedent, } |x - y| < \delta
\]
\[
= \delta|x + x + y - x| \quad \text{# add and subtract } x
\]
\[
\leq \delta(2|x| + |y - x|) \quad \text{# triangle inequality}
\]
\[
\leq \delta(2|x| + 1) \quad \text{# since } \delta \leq 1
\]
\[
< \frac{\epsilon(2|x| + 1)}{2(2|x| + 1)} \quad \text{# since } \delta \leq \epsilon/(2(2|x| + 1))
\]
\[
= \frac{\epsilon}{2} < \epsilon
\]

Then \(|x - y| < \delta \Rightarrow |x^2 - y^2| < \epsilon\). # assumed antecedent, got consequent

Then \(\forall y \in \mathbb{R}, |x - y| < \delta \Rightarrow |x^2 - y^2| < \epsilon\). # introduced \(\forall\)

Then \(\exists \delta \in \mathbb{R}^+, \forall y \in \mathbb{R}, |x - y| < \delta \Rightarrow |x^2 - y^2| < \epsilon\). # introduced \(\exists\)

Conclude \(\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, \exists \delta \in \mathbb{R}^+, \forall y \in \mathbb{R}, |x - y| < \delta \Rightarrow |x^2 - y^2| < \epsilon\). # introduced \(\forall\) twice. (ii) \(\forall x \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+, \forall x \in \mathbb{R}, |x - y| < \delta \Rightarrow |x^2 - y^2| < \epsilon\).

**Sample solution:** The claim is false. I’ll prove the negation

\[
\exists x \in \mathbb{R}^+, \forall \delta \in \mathbb{R}^+, \exists x \in \mathbb{R}, \exists y \in \mathbb{R}, |x - y| < \delta \land |x^2 - y^2| > \epsilon
\]

Pick \(\epsilon = 1\). Then \(\epsilon \in \mathbb{R}^+\). # choice to make calculations easy

Assume \(\delta \in \mathbb{R}^+\). # generic positive real number.

Pick \(x = 3/\delta\). Then \(x \in \mathbb{R}\). # since \(\delta \in \mathbb{R}^+\)

Pick \(y = x + \delta/2\). Then \(y \in \mathbb{R}\).

# since \(x, \delta, 2 \in \mathbb{R}\), and and is closed under + and division by non-zero reals

Then \(|x - y| = |x - (x + \delta/2)| = \delta/2 < \delta\). # by choice of \(y\)

Then
\[
|x^2 - y^2| = |x - y||x + y| \quad \text{# factoring inequality}
\]
\[
= \frac{\delta|x + y|}{2} \quad \text{# since } |x - y| = \delta/2
\]
\[
= \frac{\delta(2x + \delta/2)}{2} \quad \text{# since } y = x + \delta/2
\]
\[
> \frac{\delta|2x|}{2} \quad \text{# subtract positive quantity from factor}
\]
\[
= \frac{\delta|3/\delta|}{2} = 3 > 1 = \epsilon \quad \# x = 3/\delta, \text{ cancel } 2\text{s, then } \delta\text{ s}
\]

Then \(|x - y| < \delta \land |x^2 - y^2| > \epsilon\). # conjunction of results derived

Then \(\exists x \in \mathbb{R}, \exists y \in \mathbb{R}, |x - y| < \delta \land |x^2 - y^2| > \epsilon\).

# introduced \(\exists\) twice.

Then \(\forall \delta \in \mathbb{R}^+, \exists x \in \mathbb{R}, \exists y \in \mathbb{R}, |x - y| < \delta \land |x^2 - y^2| > \epsilon\).

# introduced \(\forall\)

Then \(\exists \delta \in \mathbb{R}^+, \forall \delta \in \mathbb{R}^+, \exists x \in \mathbb{R}, \exists y \in \mathbb{R}, |x - y| < \delta \land |x^2 - y^2| > \epsilon\).

# introduced \(\exists\)

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3. Use the following definition of the floor of $x$ to prove or disprove each of claims (i)–(iii).

$$\forall x \in \mathbb{R} \quad y = \lfloor x \rfloor \Leftrightarrow y \in \mathbb{Z} \land y \leq x \land (\forall z \in \mathbb{Z}, z \leq x \Rightarrow z \leq y)$$

(i) $\forall x \in \mathbb{R}, \forall z \in \mathbb{Z}, |x + z| = |x| + z$.

The claim is true. Here’s a proof.

Assume $x \in \mathbb{R}$ and $z \in \mathbb{Z}$. # generic real number, generic integer.

Let $y = \lfloor x \rfloor + z$. # convenience variable

Then $y \in \mathbb{Z}$, $\lfloor x \rfloor \in \mathbb{Z}$, by assumption and definition of $\lfloor x \rfloor$, and $\mathbb{Z}$ closed under $+$

Then $y = \lfloor x \rfloor + z \leq x + z$. # since $\lfloor x \rfloor \leq x$ by definition of $\lfloor x \rfloor$.

Then $y + 1 = \lfloor x \rfloor + 1 + z > x + z$. # from lecture $\lfloor x \rfloor + 1 > x$.

Then $\forall z' \in \mathbb{Z}, z' > y \Rightarrow z' > x + z$.

# Every integer $z'$ must satisfy $z' \geq y + 1$, from math prerequisites

Then $\lfloor x \rfloor + z = y = \lfloor x + z \rfloor$. # since $y \in \mathbb{Z} \land y \leq x + z \land (\forall z' \in \mathbb{Z}, z' > y \Rightarrow z' > x + z)$

Conclude $\forall x \in \mathbb{R}, \forall z \in \mathbb{Z}, |x + z| = |x| + z$. # introduced $\forall$ twice

(ii) $\forall x \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+, \forall \varepsilon \in [1, 2], \forall y \in [1, 2], |x - y| < \delta \Rightarrow |x^2 - y^2| < \varepsilon$.

The claim is true. Here’s the proof.

Assume $\varepsilon \in \mathbb{R}^+$. # generic positive real number.

Pick $\delta = \varepsilon / 5$. Then $\delta \in \mathbb{R}^+$.

# since $\varepsilon, 2 \in \mathbb{R}^+$, and $\mathbb{R}^+$ closed under $/$.

Assume $x \in [1, 2]$ and $y \in [1, 2]$. # generic real numbers in interval $[1, 2]$.

Assume $|x - y| < \delta$. # assume antecedent

Then

$$|x^2 - y^2| = |x - y||x + y| \quad \# \text{factor inequality}$$

$$< \delta |x + y| \quad \# \text{antecedent guarantees} |x - y| < \delta$$

$$\leq \delta (|x| + |y|) \quad \# \text{triangle inequality}$$

$$\leq 4\delta \quad \# \text{since} 1 \leq |x|, |y| \leq 2$$

$$= \frac{4\varepsilon}{5} < \varepsilon \quad \# \text{since} \delta = \varepsilon / 5$$

Then $|x - y| < \delta \Rightarrow |x^2 - y^2| < \varepsilon$. # assumed antecedent, derived consequent

Then $\forall x \in [1, 2], \forall y \in [1, 2], |x - y| < \delta \Rightarrow |x^2 - y^2| < \varepsilon$. # assumed antecedent, derived consequent

Then $\exists \delta \in \mathbb{R}^+, \forall x \in [1, 2], \forall y \in [1, 2], |x - y| < \delta \Rightarrow |x^2 - y^2| < \varepsilon$. # introduced $\exists$

Then $\forall x \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+, \forall x \in [1, 2], \forall y \in [1, 2], |x - y| < \delta \Rightarrow |x^2 - y^2| < \varepsilon$. # introduced $\forall$

3. Use the following definition of the floor of $x$ to prove or disprove each of claims (i)–(iii).
(ii) $\forall x \in \mathbb{R}, \forall z \in \mathbb{Z}, |x \times z| = |z| \times z$.

The claim is false. Here's the negation of the claim, which I then prove:

$$\exists x \in \mathbb{R}, \exists z \in \mathbb{Z}, |x \times z| \neq |z| \times z$$

Pick $x = 0.5$. Then $x \in \mathbb{R}$.

Pick $z = 2$. Then $z \in \mathbb{Z}$.

Then

$$|x \times z| = |0.5 \times 2| = 1 \quad \# \text{use values of } x \text{ and } z$$

$$\neq 0 = 0 \times 2 = |0.5| \times 2 \quad \# \text{since floor of 0.5 is 0}$$

$$= |z| \times z \quad \# \text{sub in values of } x \text{ and } z$$

Conclude $\exists x \in \mathbb{R}, \exists z \in \mathbb{Z}, |x \times z| \neq |z| \times z$. \# introduced $\exists$ twice.

(iii) $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, |x + y| \geq |x| + |y|$.

**Sample solution:** The claim is true. Here's the proof.

Assume $x \in \mathbb{R}$ and $y \in \mathbb{R}$. \# generic real numbers.

Then $|x| + |y| \in \mathbb{Z}$.

$\quad \# \text{since } |x|, |y|, \in \mathbb{Z} \text{ by definition of } |x| \text{ and } |y|, \text{ and } \mathbb{Z} \text{ closed under } +$

Then $|x| \leq x$. \# from definition of $|x|$.

Then $|y| \leq y$. \# from definition of $|y|$.

Then $|x| + |y| \leq x + y$. \# add two inequalities above

So $|x| + |y| \leq |x + y|$.

$\quad \# \text{from third clause of definition of } |x + y|, \text{ since } |x| + |y| \in \mathbb{Z} \text{ and } |x| + |y| \leq x + y$

Conclude $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, |x + y| \geq |x| + |y|$. \# introduced $\forall$ twice.