1. TRACING STATEMENTS

Consider the following sequence of natural numbers \( a_0, a_1, a_2, \ldots \):

\[
   a_n = \begin{cases} 
   0, & n \text{ is a multiple of } 4 \\
   1, & n - 1 \text{ is a multiple of } 4 \\
   2, & n - 2 \text{ is a multiple of } 4 \\
   1, & n - 3 \text{ is a multiple of } 4 
   \end{cases}
\]

Consider the following statement about (A1):

\[
   (S2) \quad \forall i \in \mathbb{N}, \exists k \in \mathbb{N}, k > i \land a_k \geq a_i.
\]

(1) For this part, let \( i = 0 \). Notice that we rewrite \( \exists k \in \mathbb{N}, k > i \land a_k \geq a_i \) as: \( \exists k \in \mathbb{N}, k > 0 \land a_k \geq 0 \).

(a) For \( k = 0 \): rewrite \( k > i \land a_k \geq a_i \), express it in English and determine whether it’s true.

(b) For \( k = 1 \): rewrite \( k > i \land a_k \geq a_i \), express it in English and determine whether it’s true.

(c) For \( k = 2 \): determine whether \( k > i \land a_k \geq a_i \) is true.

(d) Rewrite \( k > i \land a_k \geq a_i \) and express it in English.

(e) Express \( \exists k \in \mathbb{N}, k > i \land a_k \geq a_i \) in English, as naturally as possible.

(f) Determine whether \( \exists k \in \mathbb{N}, k > i \land a_k \geq a_i \) is true.

(2) Now let \( i = 1 \): rewrite \( \exists k \in \mathbb{N}, k > i \land a_k \geq a_i \).

(a) For \( k = 0 \): rewrite \( k > i \land a_k \geq a_i \), express it in English, and determine whether it’s true.

(b) For \( k = 1 \): determine whether \( k > i \land a_k \geq a_i \) is true.

(c) For \( k = 2 \): determine whether \( k > i \land a_k \geq a_i \) is true.

(d) Rewrite \( k > i \land a_k \geq a_i \) and express it in English.

(e) Express \( \exists k \in \mathbb{N}, k > i \land a_k \geq a_i \) in English, as naturally as possible.

(f) Determine whether \( \exists k \in \mathbb{N}, k > i \land a_k \geq a_i \) is true.

(3) Determine whether \( (S2) \) is true.

2. PROOF

Consider the following statement and sequence of natural numbers \( a_0, a_1, a_2, \ldots \):

\[
   (S1) \quad \exists i \in \mathbb{N}, \exists k \in \mathbb{N}, \forall m \in \mathbb{N}, m > k \implies a_m \neq a_i
\]

(A1) \( 3, 2, 1, 0, 3, 2, 1, 3, 2, 2, 2, \ldots \)

(1) Explain why \( (S1) \) is true for \( (A1) \).

(2) Give the standard structure for a proof of \( (S1) \).

(3) Fill in the structure to prove \( (S1) \) for \( (A1) \).

Consider now the sequence:

\[
   (A2) \quad 0, 1, 0, 1, 0, 1, 0, 1, \ldots
\]

(4) Explain why \( (S1) \) is false for \( (A2) \).

(5) Give the standard structure for a disproof of \( (S1) \).

(6) Fill in the structure to disprove \( (S1) \) for \( (A2) \). Use the following definition of the sequence:

\[
   \forall n \in \mathbb{N}, a_n = \begin{cases} 
   0, & n \text{ even} \\
   1, & n \text{ odd} 
   \end{cases}
\]
(2) Let $i = \_$. Then $i \in \mathbb{N}$.
Let $k = \_$. Then $k \in \mathbb{N}$.
Assume $m \in \mathbb{N}$.
Assume $m > k$.

Then $a_m \neq a_i$.
Then $m > k \Rightarrow a_m \neq a_i$
Then $\forall m \in \mathbb{N}, m > k \Rightarrow a_m \neq a_i$
Then $\exists i \in \mathbb{N}, \exists k \in \mathbb{N}, \forall m \in \mathbb{N}, m > k \Rightarrow a_m \neq a_i$

(3) set $i = 0$,
set $k = 7$,
replace “…” with:
Then $m \geq 8$
Then $a_m = 2 \neq 3 = a_0 = a_i$.

(5) After negating:
Assume $i \in \mathbb{N}$.
Assume $k \in \mathbb{N}$.
Let $m = \_$. Then $m \in \mathbb{N}$.

Then $m > k$.

Then $a_m = a_i$.
Then $\exists m \in \mathbb{N}, m > i \wedge a_m = a_i$.
Then $\forall i \in \mathbb{N}, \forall k \in \mathbb{N}, \exists m \in \mathbb{N}, m > k \wedge a_m = a_i$.

(6) Replace “__” with: $2k + 2 + i$
First “…”: Then $2k + 2 + i > 2k + i \geq 2k \geq k$
Second “…”: Then $2k + 2 + i$ is even iff $i$ is even

You could explore cases:
based on $i$, to prove that last ...
based on $i$ and $k$, to pick $m$