Please read the following guidelines carefully!

- Please write your name on both the front and back of this exam.
- This examination has 4 questions. There are a total of 8 pages, DOUBLE-SIDED.
- Answer questions clearly and completely. Provide justification unless explicitly asked not to.
- All formulas must have negations applied directly to propositional variables or predicates.
- In your proofs, you may always use definitions of predicates from the course. You may not use any external facts about rates of growth, divisibility, primes, or greatest common divisor unless you prove them, or they are given to you in the question.

Take a deep breath.
This is your chance to show us
How much you’ve learned.

Good luck!
1. [6 marks] Statements in logic.

   (a) [3 marks] Write the truth table for the following formula. No rough work is required.

   
   \[(p \Rightarrow q) \lor \neg r) \iff \neg p\]

   **Solution**

<table>
<thead>
<tr>
<th>p</th>
<th>q</th>
<th>r</th>
<th>((p \Rightarrow q) \lor \neg r) \iff \neg p\</th>
</tr>
</thead>
<tbody>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>F</td>
</tr>
</tbody>
</table>

   (b) [3 marks] Consider the pair of statements:

   
   \( (1) \ \exists n \in \mathbb{N}, \ P(n) \iff Q(n) \) \hspace{1cm} \( (2) \ \exists n \in \mathbb{N}, \ P(n) \land Q(n) \)

   Define the predicates \( P \) and \( Q \) with domain \( \mathbb{N} \) so that one of these statements is true and the other one false. Note that you’re only defining the predicates once: the two statements must use the same definitions for \( P \) and \( Q \). Also, briefly explain which statement is true and which one is false, and why; no formal proofs necessary.

   **Solution**

   Let \( P(n) \) be the predicate “\( n < 0 \)” and \( Q(n) \) be the predicate “\( n < -2 \)”.

   The first statement becomes \( \exists n \in \mathbb{N}, n < 0 \iff n < -2 \), which is true for 0, since false \( \iff \) false. The second statement becomes \( \exists n \in \mathbb{N}, n < 0 \land n < -2 \), which is false for every natural number.
2. [7 marks] Translating statements.

A powerful number is a positive integer $m$ such that for every prime $p$ that divides $m$, $p^2$ also divides $m$.

Express each of the following statements using predicate logic. No justification is required. Note: please review the instructions on the midterm’s front page for our expectations in this question. In particular, you may not define any helper predicates or sets.

(a) [2 marks] 24 is not a powerful number.

Solution

$$\exists p \in \mathbb{N}, p \mid 24 \land \text{Prime}(p) \land p^2 \nmid 24$$

(b) [5 marks] 81 is the smallest powerful number greater than 72.

Solution

$$[\forall p \in \mathbb{N}, (p \mid 81 \land \text{Prime}(p)) \Rightarrow p^2 \mid 81] \land \forall n \in \mathbb{N}, [n < 81 \land \forall p_0 \in \mathbb{N}, (p_0 \mid n \land \text{Prime}(p_0)) \Rightarrow p_0^2 \mid n] \Rightarrow n < 73$$
3. [6 marks] Proofs (inequalities). Consider the following statement: “For every natural number $x$ there is a natural number $y$ such that $15 > xy > 5$.”

(a) [1 mark] Translate the above statement into predicate logic. Use the symbol to denote the set of positive real numbers.

Solution

\[ \forall x \in \mathbb{N}, \exists y \in \mathbb{N}, 15 > xy \land xy > 5 \]

(b) [1 mark] Write the negation of this statement, fully-simplified so that all negation symbols are applied directly to predicates. You can simplify $\neg(a > b)$ to $a \leq b$.

Solution

\[ \exists x \in \mathbb{N}, \forall y \in \mathbb{N}, 15 \leq xy \lor xy \leq 5 \]

(c) [4 marks] Disprove the original statement by proving its negation. In your proof, any chains of calculations must follow a top-down order; don’t start with the inequality you’re trying to prove!

Write any rough work or intuition in the Discussion box, and write your formal proof in the Proof box. Your rough work/intuition will only be looked at if your proof is not completely correct.

Solution

Proof. Let $x = 15$. Let $y \in \mathbb{N}$. I need to prove that either $xy \geq 15$ or $xy \leq 5$.

There are two cases to consider.

Case $y \geq 1$: Then $xy \geq 1 \times x = 15 \geq 15$

Case $y = 0$: Then $xy = 0 \leq 5$. 

\[ \square \]
4. [5 marks] Proofs (number theory). Consider the following statement: “If \( m \) and \( n \) are integers, and 5 divides both \( m \) and \( n \), then 5 divides \( 2m + n \).”

(a) [1 mark] Translate the above statement into predicate logic.

\[
\forall m, n \in \mathbb{Z}, (5 \mid m \land 5 \mid n) \Rightarrow 5 \mid (2m + n)
\]

Solution

(b) [4 marks] Prove the above statement using the definition of divisibility:

\[ x \mid y : \exists k \in \mathbb{Z}, y = kx \]

Do not use any external facts about divisibility.

Write any rough work or intuition in the Discussion box, and write your formal proof in the Proof box. Your rough work/intuition will only be looked at if your proof is not completely correct.

Solution

Proof. Let \( m, n \in \mathbb{Z} \). Assume \( 5 \mid m \land 5 \mid n \), that is \( \exists k_1, k_2 \in \mathbb{Z}, m = 5k_1 \land n = 5k_2 \). Let \( k_1, k_2 \) be such values. Let \( k_3 = 2k_1 + k_2 \). I need to show that \( 2m + n = 5k_3 \).

Then,

\[
2m + n = 2(5k_1) + 5k_2 = 5(2k_1 + k_2) = 5k_3.
\]
This page is left nearly blank for rough work. If you want work on this page to be marked, please indicate this clearly at the location of the original question.
This page is left nearly blank for rough work. If you want work on this page to be marked, please indicate this clearly at the location of the original question.
Name:

<table>
<thead>
<tr>
<th>Question</th>
<th>Grade</th>
<th>Out of</th>
</tr>
</thead>
<tbody>
<tr>
<td>Q1</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>Q2</td>
<td>7</td>
<td></td>
</tr>
<tr>
<td>Q3</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>Q4</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>24</strong></td>
<td></td>
</tr>
</tbody>
</table>