Learning Objectives

By the end of this worksheet, you will:

- Prove and disprove statements using the definition of Big-Oh.
- Investigate properties of Big-Oh of some common functions.

**Note:** in Big-Oh expressions, it will be convenient to just write down the “body” of the functions rather than defining named functions all the time. We’ll always use the variable \( n \) to represent the function input, and so when we write “\( n \in \mathcal{O}(n^2) \),” we really mean “the functions defined as \( f(n) = n \) and \( g(n) = n^2 \) satisfy \( f \in \mathcal{O}(g) \).”

As a reminder, here is the formal definition of what “\( g \) is Big-Oh of \( f \)” means:

\[
g \in \mathcal{O}(f) : \exists c, n_0 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow g(n) \leq cf(n)
\]

1. **Comparing polynomials.** Our first step in comparing different types of functions is looking at different powers of \( n \). Consider the following statement, which generalizes the idea that \( n \in \mathcal{O}(n^2) \):

\[
\forall a, b \in \mathbb{R}^+, \ a \leq b \Rightarrow n^a \in \mathcal{O}(n^b)
\]

(a) Rewrite the above statement, but with the definition of Big-Oh expanded.

(b) Prove the above statement. **Hint:** you can actually pick \( c \) and \( n_0 \) to both be 1, and have the proof work.
2. **Comparing logarithms.** One slight oddness about the definition of Big-Oh is that it treats all logarithmic functions “the same.” Your task in this question is to investigate this, by proving the following statement:

\[
\forall a, b \in \mathbb{R}^+, \ a > 1 \land b > 1 \Rightarrow \log_a n \in O(\log_b n)
\]

We won’t ask you to expand the definition of Big-Oh, but if you aren’t quite sure, then we highly recommend doing so before attempting even your rough work!

**Hint:** look up the “change of base rule” for logarithms, if you don’t quite remember it!
3. **Sum of functions.** Now let's look at one of the most important properties of Big-Oh: how it behaves when adding functions together. Let \( f, g : \mathbb{N} \to \mathbb{R}^{\geq 0} \) (i.e., \( f \) and \( g \) are two functions that take natural numbers and return non-negative real numbers). We can define the sum of \( f \) and \( g \) as the function \( f + g : \mathbb{N} \to \mathbb{R}^{\geq 0} \) such that

\[
\forall n \in \mathbb{N}, \ (f + g)(n) = f(n) + g(n).
\]

For example, if \( f(n) = 2n \) and \( g(n) = n^2 + 3 \), then \( (f + g)(n) = 2n + n^2 + 3 \).

Consider the following statement:

\[
\forall f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}, \ g \in \mathcal{O}(f) \Rightarrow f + g \in \mathcal{O}(f)
\]

In other words, if \( g \) is Big-Oh of \( f \), then \( f + g \) is no bigger than just \( f \) itself, asymptotically speaking.\(^1\)

Your task for this question is to prove this statement. Keep in mind this is an implication: you're going to assume that \( g \in \mathcal{O}(f) \), and you want to prove that \( f + g \in \mathcal{O}(f) \). It will likely be helpful to write out the full statement (with the definition of Big-Oh expanded), and use subscripts to help keep track of the variables.

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\(^1\) This statement is quite similar to ones about divisibility, and in particular Question 1 on Problem Set 2.