CSC165 fall 2017
graph connectivity, trees

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Using Course notes: average analysis; graphs
Outline

notes
must be connected

\[ \forall n \in \mathbb{N}, \exists M \in \mathbb{N}, \forall G = (V, E), (V = n \land E \geq M) \Rightarrow G \text{ is connected?} \]

\[ \frac{n(n-1)}{2} = |E| \]

\[ \frac{(n-1)(n-2)}{2} \]

\[ \forall n \in \mathbb{N}^+, \forall G = (V, E), (|V| = n \land |E| \geq \frac{(n-1)(n-2)}{2} + 1) \Rightarrow G \text{ is connected} \]

Proof (induction on n).

- **Base case**: vacuously true—no graph with 1 vertex and \( \geq 1 \) edge
must be connected

\[ \forall n \in \mathbb{N}, \exists M \in \mathbb{N}, \forall G = (V, E), (V = n \land E \geq M) \Rightarrow G \text{ is connected} \]

\[ p(n) \quad \ldots \quad \frac{(n-1)(n-2)}{2} + 1 \]

Inductive step: Let \( k \in \mathbb{N}^+ \). Assume \( p(k) \). Want to show \( p(k+1) \) must follow.

Don't do this!!

Start with an graph on \( k \) vertices with \( \frac{(k-1)(k-2)}{2} + 1 \) edges. By IH it is connected. Add \( k-1 \) edges + vertex, result connected. Induction trap

Do this: Let \( G = (V, E) \) be an arbitrary graph with \( |V| = k+1 \) and \( |E| = \frac{k(k-1)}{2} + 1 \).

Trap II: Choose \( u \in V \), let \( G' = (V', E') \) where \( V' = V \setminus \{u\}, E' = E \setminus \{u,v\} \} \).
must be connected

\( \forall n \in \mathbb{N}, \exists M \in \mathbb{N}, \forall G = (V, E), (V = n \land E \geq M) \Rightarrow G \) is connected?

Then \( |V'| = k \), \( |E'| \geq \frac{k(k-1)}{2} + 1 - k \)

\[
= \frac{k(k-1)}{2} - \frac{2k}{2} + 1
\]

\[
= \frac{k(k-3)}{2} + 1 = \frac{k^2 - 3k + 2}{2}
\]

\[
\leq \frac{k^2 - 3k + 2}{2} + 1 = \frac{(k-1)(k-2)}{2} + 1
\]

Case 1: \( |E| = \frac{k(k+1)}{2} \), so \( G \) is complete \& thus connected.

Case 2: \( |E| < \frac{k(k+1)}{2} \) Need vertex \( u \) with \( 1 \leq d(u) \leq k \).
∀n ∈ N, \exists G = (V, E), V = n ∧ E = n - 1 ∧ G is connected

Then d(u) > 0. (otherwise G - u has ≤ \frac{R(R-1)}{2} \rightarrow \leftarrow)

Also, if G is not complete \exists u, v \in V \ s.t. (u, v) \notin E

Choose u, note d(u) ≤ R - 1.

Let G' = (V', E'), V' = V \setminus \{u\}, E' = E \setminus \{(u, v) | (u, v) \in E\}

Then |V'| = n and \quad |E'| ≥ |E| - (R - 1)

\quad \geq \frac{R(R-1)}{2} + 1 - (R - 1)

\quad \geq \frac{R(R-1)}{2} + \frac{1}{2}

\quad = \frac{R(R-1)}{2} - 2(R - 1) + 1

\quad = \frac{R(R-1)}{2} - \frac{2(R-1)}{2} + 1

\quad = \frac{(R-1)(R-2)}{2} + 1

Thus, by IH, G' is connected.
maybe be connected

\[ \forall n \in \mathbb{N}, \exists G = (V, E), V = n \land E = n - 1 \land G \text{ is connected} \]

\[ \text{\underline{\(- P_n \)}} \quad \text{path on n vertices - is connected} \]

\[ \begin{array}{c}
\text{by n-1 edges} \\
\end{array} \]

\[ \text{Case study} \ G = \text{Facebook} \]

\[ |V| = 2B \ (?) \]

\[ \frac{(2B-1)(2B-2)}{2} + 1 \rightarrow G \text{ connected} \]

\[ \frac{2B-1}{2} \text{ friendships} \]

\[ \text{if controlled which friends} \]
maybe be connected

\( \forall n \in \mathbb{N}, \exists G = (V, E), V = n \wedge E = n - 1 \wedge G \) is connected

Since \( d(u) \geq 1 \), u has/had neighbour \( w \in V \)

Thus u connected to w + w connect to all other vertices \( \implies G \) connected.
must be disconnected

\[ \forall n \in \mathbb{N}, \forall G = (V, E), (V = n \land E \leq n - 2) \Rightarrow G \text{ is not connected} \]

steps:

- natural to reason by removing an edge from a connected graph with \( n - 1 \) edges...
- first need some results about which components of connected graphs have redundant edges (cycles)...
- then need some results about connected graphs without cycles (trees)...
- then reason about reducing an arbitrary connected graph to a tree... (by pruning)

whew!
cycle

consecutively adjacent vertices \(v_0, \ldots, v_k \in V \land k \geq 3\),
all distinct except \(v_0 = v_k\)

\(\forall G = (V, E), \forall e \in E, G \text{ connected } \Rightarrow (e \text{ in a cycle of } G \iff G - e \text{ connected })\)

**Proof**

Let \(G = (V, E)\), let \(e \in E\), assume \(G\) is connected. Assume \(e\) in a cycle.

Let \(w_1, w_2 \in V\). Since \(G\) is connected, there is a path, \(P\), from \(w_1\) to \(w_2\).

**Case 1\)** \(e = (u, v)\) not in \(P\). Then \(P\) is a path in \(G - e\).

\(w_1, w_2\) are connected in \(G - e\).

**Case 2** Path \(P\) includes edge \(e = (u, v)\). Let \(P_1\) be the path from \(w_1\) to nearest of \(u, v\) in \(P\).

Let \(P_2\) be the path from \(w_2\) to nearest of \(u, v\) in \(P\). Since \(e = (u, v)\) is in a cycle, there is a path from \(u\) to \(v\) after \(e = (u, v)\) removed.
cycle

consecutively adjacent vertices \( v_0, \ldots, v_k \in V \land k \geq 3 \),
all distinct except \( v_0 = v_k \)

\( \forall G = (V, E), \forall e \in E, G \text{ connected } \Rightarrow (e \text{ in a cycle of } G \Leftrightarrow G - e \text{ connected}) \)

the \( P_1 \cup P_2 \cup P_3 \) is a path \( u_1 \) to \( u_2 \) in \( G - e \).

Proof\\( \leq \) Idea if \( e = (u, v) \) and \( G - e \) connected
then \( \exists \text{ Path from } u \text{ to } v \) in \( G - e \).

Then show Path \( U(c, u, v) \) is a cycle
tree: connected, acyclic graph

removing any edge from a tree disconnects it

\[ \forall G = (V, E), \, G \text{ is a tree } \Rightarrow E = V - 1 \]

but first...

\[ \forall G = (V, E), (G \text{ is a tree } \land V \geq 2) \Rightarrow (\exists v \in V, d(v) = 1) \]

reason want to let \( G = (V, E) \) be an arbitrary tree \(|V| = n+1\) vertices
- remove 1 vertex (and 1 edge)

on use IH
tree: connected, acyclic graph

removing any edge from a tree disconnects it

∀G = (V, E), G is a tree ⇒ E = V − 1

but first...

∀G = (V, E), (G is a tree ∧ V ≥ 2) ⇒ (∃v ∈ V, d(v) = 1)
main result...

∀n ∈ \mathbb{N}^+, \forall G = (V, E), (G is a tree \land V = n) \Rightarrow E = V - 1

P(n):

Base case: The only tree with \(|V| = 1\) is a graph with 0 = 1 - 1 edges.

Inductive step: Let n ∈ \mathbb{N}^+. Assume P(k) \forall k \in \mathbb{N}^+, k ≤ n. i.e. every tree with k ≤ n vertices has exactly k-1 edges. Want to prove P(n+1): \forall k \in \mathbb{N}, k ≤ n+1, tree with k vertices has k-1 edges.

Let T = (V, E) be a tree with \(|V| = n+1\), T is disconnected.

Remove edge e = (u, v) from E. T is disconnected.

Let V_i \subseteq V be vertices that were connected to u without using v.
main result...

\[ \forall n \in \mathbb{N}^+, \forall G = (V, E), (G \text{ is a tree } \land V = n) \Rightarrow E = V - 1 \]

Let \( V_2 = V \setminus V_1 \). \( V_1 \cap V_2 = \emptyset \) (due to no cycles).

and \( V_1 \cup V_2 = V \).

\( T_1 = (V_1, E_1) \) \( T_2 = (V_2, E_2) \)

\( |E_1| = |V_1| - 1 \) \( \square \) and \( |E_2| = |V_2| - 1 \)

so \( |E| = |E_2| + |E_1| + 1 \)

\[ = |V_1| - 1 + |V_2| - 1 + 1 \]

\[ = |V_1| + |V_2| - 1 = (|V| - 1) \]
big picture...

\[ |E| = \frac{n(n-1)}{2} \]

must be connected

\[ \frac{(n-1)(n-2)}{2} \]

must be disconnected

\[ |E| = 0 \]

slightly alter parameters by minimum specifying maximum degree