CSC165 fall 2014
Mathematical expression

Danny Heap
heap@cs.toronto.edu
BA4270 (behind elevators)
http://www.cdf.toronto.edu/~heap/165/F14/
416-978-5899

Course notes, chapter 3
Outline

universally quantified implication, cont’d

existence

notes

annotated slides
Prove that for every pair of non-negative real numbers \((x, y)\), if \(x\) is greater than \(y\), then the geometric mean, \(\sqrt{xy}\) is less than the arithmetic mean, \((x + y)/2\).
Prove that for any natural number $n$, $n^2$ odd implies that $n$ is odd.
To prove the a set is non-empty, it’s enough to exhibit one element. How do you prove:

\[ \exists x \in \mathbb{R}, x^3 + 3x^2 - 4x = 12 \]

**Proof**

Pick \( x = 2 \). Then \( x \in \mathbb{R} \) is well-known.

Then \( x^3 + 3x^2 - 4x = 8 + 12 - 8 = 12 \) # Sub 2 for \( x \)

Then \( \exists x \in \mathbb{R}, x^3 - 3x^2 - 4x = 12 \) # 2 \( \in \mathbb{R} \) satisfies eqn.
prove a claim about a sequence

Define sequence \( a_n \) by:

\[
\sqrt{x} < \sqrt{y} \Rightarrow \sqrt{x} < \sqrt{y} \quad \forall n \in \mathbb{N} \quad a_n = n^2
\]

Now prove:

\[
\exists i \in \mathbb{N}, \forall j \in \mathbb{N}, a_j \leq i \Rightarrow j < i
\]

**Proof**

Pick \( i = 2 \). Then \( i \in \mathbb{N} \neq 2 \in \mathbb{N} \).

Assume \( j \) is a representative of \( \mathbb{N} \).

Assume \( a_j \leq i \neq \) assume \( A \).

Then \( j^2 \leq 2 \) # by defn \( a_j + i = 2 \).

Then \( j \leq \sqrt{2} \) # because \( \sqrt{A} \) necessary.

So \( j \leq \sqrt{2} < 2 \neq \sqrt{2} \times 1.414 \)
prove a claim about a sequence

Define sequence \( a_n \) by:

\[
\forall n \in \mathbb{N} \quad a_n = n^2
\]

Now prove:

\[
\exists i \in \mathbb{N}, \forall j \in \mathbb{N}, a_j \leq i \Rightarrow j < i
\]
contradiction — a special case of contrapositive

\[ F_1 \land F_2 \land \cdots \land F_{367893221596} \Rightarrow S \]

\[ \neg S \Rightarrow \neg F_1 \lor \neg F_2 \lor \cdots \lor \neg F_{367893221596} \]

Define the prime natural numbers as

\[ P = \{ p \in \mathbb{N} \mid p \text{ has exactly two distinct divisors in } \mathbb{N} \} \]. How do you prove:

\[ S : \forall n \in \mathbb{N}, |P| > n \]

It would be nice to have some result \( R \) that leads to \( S \). If you could show \( R \Rightarrow S \), and that \( R \) is true, then you’d be done. But, out of many elementary results, how do you choose an \( R \)? Contradiction will often lead you there.

\[ \neg S \quad \exists n \in \mathbb{N}, |P| \leq n \]
non-boolean functions

Take care when expressing a proof about a function that returns a non-boolean value, such as a number:

\[ \lfloor x \rfloor \text{ is the largest integer } \leq x. \]

Now prove the following statement (notice that we quantify over \( x \in \mathbb{R} \), not \( \lfloor x \rfloor \in \mathbb{R} \):

\[ \forall x \in \mathbb{R}, \lfloor x \rfloor < x + 1 \]
You may have been disappointed that the last proof used only part of the definition of floor. Here’s a symbolic re-writing of the definition of floor:

\[ \forall x \in \mathbb{R} \quad y = \lfloor x \rfloor \iff y \in \mathbb{Z} \land y \leq x \land (\forall z \in \mathbb{Z}, z \leq x \implies z \leq y) \]

The full version of the definition should prove useful to prove:

\[ \forall x \in \mathbb{R}, \lfloor x \rfloor > x - 1 \]
proving something false

Define a sequence:

$$\forall n \in \mathbb{N} \quad a_n = \lfloor n/2 \rfloor$$

(of course, if you treat “/” as integer division, there’s no need to take the floor. Now consider the claim:

$$\exists i \in \mathbb{N}, \forall j \in \mathbb{N}, j > i \Rightarrow a_j = a_i$$

The claim is false. Disprove it.
Sometimes your argument has to split to take into account possible properties of your generic element:

\[ \forall n \in \mathbb{N}, n^2 + n \text{ is even} \]

A natural approach is to factor \( n^2 + n \) as \( n(n + 1) \), and then consider the case where \( n \) is odd, then the case where \( n \) is even.
Proof (Contradiction)

Assume \( \exists n \in \mathbb{N}, \ |P| \leq n \)

Then \( \exists k \in \{0, 1, \ldots, n^2\}, \ |P| = k \)

Then \( \{p_1, p_2, \ldots, p_k\} = P \) \# just list primes

Then \( m = p_1 \times p_2 \times \ldots \times p_k \in \mathbb{N} \)

Then \( m + 1 \in \mathbb{N} \). \# \mathbb{N} closed under \( \times, + \)

Then \( m + 1 > 1 \) \# \( m \geq 2 \times 3 \times 5 \times \ldots \)

Then \( \exists p \in P, \ p \mid (m+1) \) \# every \( n \in \mathbb{N}, n > 1 \) has prime factor.

Also \( p \mid m \) \# since \( m = p_1 \times p_2 \times \ldots \times p_k \)

So \( p \mid (m+1 - m) = 1 \) \# factor divides difference

Then \( p \mid 1 \rightarrow \text{contradiction!!} \)

So, our assumption that \( \exists n \in \mathbb{N}, \ |P| \leq n \) is false, since it leads to a contradiction.
annotated slides

- Monday’s annotated slides
- Wednesday’s annotated slides
- Friday’s annotated slides