Linear Filters and Convolution

Ahmed Ashraf
The Linear Filters that we are studying in the course belong to a class of systems known as Linear Time Invariant (LTI) systems. Traditionally they have been studied for signals that vary with time. But since we are currently dealing with single images, which vary with space, such filters for images are termed as Linear Shift Invariant (LSI) Filters.
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Linear Time(Shift) Invariant (LTI) Systems

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**Linearity**: Linearity is a combination of two properties in turn:

i) If \( f(t) \) produces an output \( g(t) \), then for a linear system, \( a.f(t) \) produces an output \( a.g(t) \) 

*(This property is called HOMOGENEITY)*

ii) If \( f_1(t) \) produces \( g_1(t) \), AND \( f_2(t) \) produces \( g_2(t) \), then for a linear system, \( f_1(t) + f_2(t) \) produces \( g_1(t) + g_2(t) \).

*(This property is called SUPERPOSITION)*
Impulse Functions

Impulse Centered at the Origin
Impulse Functions

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$\delta(t)$

Shifted Impulse (Right Shifted)

$\delta(t-3)$
Impulse Functions

Impulse Centered at the Origin

Shifted Impulse (Right Shifted)

Shifted Impulse (Left Shifted)
Impulse Functions

To figure out the direction of the shift, imagine this as a shift in the origin, and see where the argument of the impulse function is zero, i.e. $t-3$ is zero for $t=+3$, and $t+3$ is zero for $t=-3$. 

- **Impulse Centered at the Origin**

- **Shifted Impulse (Right Shifted)**

- **Shifted Impulse (Left Shifted)**
Arbitrary Function as a weighted sum of shifted impulses

\[ f(t) = \sum f(t_{i}) \delta(t - t_{i}) \]
Impulse Response

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Because, if we know the response to an impulse, we can use shift invariance, and linearity (homogeneity and superposition) to compute the output to arbitrary inputs.
If the impulse response of a system (filter) is $h(t)$, i.e. it produces an output $h(t)$ in response to $\delta(t)$, what would be its output in response to the following function as an input:

$$f(t) = \sum_{i} f(t_i) \delta(t - t_i)$$
Given Impulse Response, Computing output to arbitrary function

• If the impulse response of a system (filter) is \( h(t) \), i.e. it produces an output \( h(t) \) in response to \( \delta(t) \), what would be its output in response to the following function as an input:

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f(t) = \sum_{i} f(t_i) \delta(t - t_i)
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• Given: \( \delta(t) \) produces \( h(t) \)
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- Given:

$\delta(t)$ produces $h(t)$

$\Rightarrow\delta(t - t_i)$ produces $h(t - t_i)$ (Using Shift Invariance)
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  \[\Rightarrow f(t_i)\delta(t - t_i) \text{ produces } f(t_i)h(t - t_i) \quad \text{(Using Homogeneity)}\]
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\Rightarrow \quad f(t) \text{ produces } \sum_i f(t_i)h(t - t_i) \quad \text{(Using Superposition)}
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Given Impulse Response, Computing output to arbitrary function

- If the impulse response of a system (filter) is $h(t)$, i.e. it produces an output $h(t)$ in response to $\delta(t)$, what would be its output in response to the following function as an input:

$$f(t) = \sum_i f(t_i)\delta(t - t_i)$$

- Given:
  - $\delta(t)$ produces $h(t)$
  - $\Rightarrow \delta(t - t_i)$ produces $h(t - t_i)$ (Using Shift Invariance)
  - $\Rightarrow f(t_i)\delta(t - t_i)$ produces $f(t_i)h(t - t_i)$ (Using Homogeneity)
  - $\Rightarrow f(t)$ produces $\sum_i f(t_i)h(t - t_i)$ (Using Superposition)

This sum is the convolution of $f(t)$ and $h(t)$, written as $f(t)*h(t)$, and IS the output of the filter, i.e. the output is the correlation of the flipped $h(t)$ with $f(t)$.
Convolution between $f(t)$ and $h(t)$

$$f(t) * h(t) = \sum_{i} f(t_i) h(t - t_i)$$

- *i.e., the concepts of convolution, flipping one signal, and then taking its correlation with the input to get the output are NOT handed to us by fiat. Rather they naturally emerge from the properties of Linearity and Time/Shift Invariance.*

- **Homework:** Extend these concepts for 2D images and material covered in Lecture 3.