Search

- Chapter 3 of R&N* 3rd edition is very useful reading.
- Chapter 4 of R&N 3rd edition is worth reading for enrichment. We'll touch upon some of the ideas again when we look at planning.

* R&N = Russell and Norvig, Artificial Intelligence: a Modern Approach

Example: Holiday Planning

We must take into account various preferences and constraints to develop a plan.

An important technique in developing such a plan is "hypothetical" reasoning.

Example: a holiday in British Columbia
- e.g., if I fly into Vancouver at 8pm and drive a car to Whistler, I'll have to drive on the roads at night. How desirable is this?
- If I'm in Whistler and leave at 7:30am, I can arrive in Kamloops in time for lunch.

How do we plan our holiday?

This kind of hypothetical reasoning involves asking
"what state will I be in after the following sequence of events?"

From this we can reason about what sequence of events one should try to bring about to achieve a desirable state.

Search is a computational method for capturing a particular version of this kind of reasoning.

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What is Search

- One of the most fundamental techniques in AI
  - Underlying sub-module in many AI systems.
- Solves many problems that humans are not good at.
- Can achieving super-human performance on other problems (Chess, Go)
- Very useful as a general algorithmic technique for solving problems (both in AI and in other areas)

Diversity of Applications

Search Problems

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"what state will I be in after the following sequence of events?"

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Search is a computational method for capturing a particular version of this kind of reasoning.
Search Problems

Diversity of Applications

Successful
• Many AI problems can be successfully solved by search
• Many AI search programs outperform humans (e.g., games)

Practical
• Many problems don’t have a simple algorithmic solution.
  Casting these problems as search problems is often the easiest
  way of solving them.
• Search can also be useful in approximation (e.g., local search in
  optimization problems).
• Problem-specific knowledge can be exploited via heuristics.

Why Search

Some critical aspects of intelligent behaviour, e.g., planning, can be
naturally cast as search.

Challenge

Problem → Search Problem

Search only shows how to solve the problem once we have it correctly formulated.

The Formalism

To formulate a problem as a search problem we need the following components:
1. a state space over which to search. The state space
  necessarily involves abstracting the real problem.
2. actions (successor functions, state space transitions) that
  allow one to move between different states. The actions
  may be abstractions of actions you could actually perform.
3. the initial state that best represents the current state .
4. the goal or desired condition one wants to achieve.
5. (optionally) costs that are associated with actions.
6. (optionally) heuristics to help guide the search process.

Utilizing the formalism to search

Once the problem has been formulated as a state space search,
various algorithms can be utilized to solve the problem.

A solution to the problem will be a sequence of actions/moves
that can transform your current state into state where your
desired (goal) condition holds.

Example 1: Romania Travel.

Currently in Arad, need to get to Bucharest by tomorrow to
catch a flight. What is the State Space?

McIlraith & Allin, CSC384 Winter 2018
**Example 1**

**State Space**
- **States:**
  - the cities you could be located in.
- **Actions:**
  - drive between neighboring cities.
- **Initial state:**
  - in Arad
- **Desired condition (Goal):**
  - be in a state where you are in Bucharest. (How many states satisfy this condition?)

**Solution**
- route, the sequence of cities to travel through to reach Bucharest.

**Example 2. The 8-Puzzle**

**State space**
- **States:** the different configurations of the tiles.
  - How many different states?
- **Actions (successor function):** moving the blank up, down, left, right. Can every action be performed in every state?
  - **Initial state:** as shown on previous slide.
- **Desired condition (Goal):** a state where the tiles are all in the positions shown on the previous slide.

**Solution**
- will be a sequence of moves of the blank that transform the initial state to a goal state.

**Example 2. The 8-Puzzle**

**Rule:** Can slide a tile into the blank spot.
  - (Equivalently, can think of it as moving the blank around).

**Example Search Space**

**Example 2. The 8-Puzzle**

Although there are 9! different configurations of the tiles (362,880) in fact the state space is divided into two disjoint parts.
- Only when the blank is in the middle are all four actions possible.
- Our goal condition is satisfied by only a single state.
- But one could easily have a goal condition like
  - The 8 is in the upper left hand corner.
  - How many different states satisfy this goal?

**Example 3. Vacuum World**

In the previous two examples, a state in the search space corresponded to a unique state of the world (modulo details we have abstracted away).

However, states need not map directly to world configurations. Instead, a state could map to the agent’s mental conception of how the world is configured: the agent’s knowledge state.
Example 3. Vacuum World

- We have a vacuum cleaner and two rooms.
- Each room may or may not be dirty.
- The vacuum cleaner can move left or right (the action has no effect if there is no room to the right/left).
- The vacuum cleaner can suck; this cleans the room (even if the room was already clean).

Physical states

Goal is to have all rooms clean.

Knowledge-level State Space

- The state space can consist of a set of states. The agent knows that it is in one of these states, but doesn’t know which.

Goal is to have all rooms clean.

Knowledge-level State Space

- Complete knowledge of the world: agent knows exactly which state it is in. State space states consist of single physical states:
  - Start in (5):
    - <right, suck>

Goal is to have all rooms clean.

Initial state:

(1,2,3,4,5,6,7,8)

Left

Suck

McIlraith & Allin, CSC384 Winter 2018
Example 3. Vacuum World

More complex situations

Instead of complete lack of knowledge, the agent might think that some states of the world are more likely than others.
This leads to probabilistic models of the search space and different algorithms for solving the problem.
Later we will see some techniques for reasoning and making decisions under uncertainty.

We can also consider the outcome of an action to be nondeterministic, perhaps associating a probability with each outcome.

Diversity of Applications

ALGORITHMS FOR SEARCH
Algorithms for Search

Inputs:
- a specified initial state (a specific world state or a set of world states representing the agent’s knowledge, etc.)
- a successor function $S(x)$ (set of states that can be reached from state $x$ via a single action).
- a goal test a function that can be applied to a state and returns true if the state is satisfies the goal condition.
- (optionally) a step cost function $C(x,a,y)$ which determines the cost of moving from state $x$ to state $y$ using action $a$. $C(x,a,y) = \infty$ if $a$ does not yield $y$ from $x$. By default this is “1” for all actions (uniform cost).

Output:
- a sequence of states leading from the initial state to a state satisfying the goal test.
- The sequence might be:
  - annotated by the name of the action used.
  - optimal in cost for some algorithms.

Tree search

- We put nodes (or states) we haven’t explored or “expanded”, but wish to consider, in a list called the Frontier (or Open list)
- Initially, the Frontier contains the initial state
- At each search iteration, we pop a node from the Frontier, apply Successors($x$), and insert those children into the Frontier

TreeSearch(Frontier, Successors, Goal)

- If Frontier is empty return failure
- Curr – select state from Frontier
- If (Goal(Curr)) return Curr.
- Frontier’ = (Frontier – {Curr}) U Successors(Curr)
- return TreeSearch(Frontier’, Successors, Goal)

Solution:
- Arad -> Sibiu -> Fagaras -> Bucharest
  Cost: $140 + 80 + 211 = 431$

Solution:
- Arad -> Sibiu -> Rimnicu Vlaicu -> Piatra Neamt -> Bucharest
  Cost: $140 + 80 + 97 + 101 = 418$

To explore the state space during a search, we will iteratively apply the successor function to the states we discover.
- Each time, the successor function $S(x)$ yields a set of states that can be reached from $x$ via any single action.
- Rather than just return a set of states, we might annotate these states by the action used to obtain them:
  - $S(x) = \{y_1, a_1\}$ arrive at $y_1$ via action $a_1$.
  - $S(x) = \{y_1, a_1, y_2, a_2\}$ arrive at $y_1$ via action $a_1$, also $y_2$ via alternative action $a_2$.
- It may also be important to reference the state of origin (i.e., the preceding state).
Frontier is a set of paths not a set of states: cycles become an issue.

Partial Search Space

Selection Rule

The example shows that the order states are selected from the frontier has a critical effect on the operation of the search:

• Whether or not a solution is found
• The cost of the solution found.
• The time and space required by the search.

Critical Properties of Search

Completeness: will the search always find a solution if a solution exists?
Optimality: will the search always find the least cost solution? (when actions have costs)
Time complexity: what is the maximum number of nodes than can be expanded or generated?
Space complexity: what is the maximum number of nodes that have to be stored in memory?

UNINFORMED SEARCH STRATEGIES

Uninformed Search Strategies

These are strategies that adopt a fixed rule for selecting the next state to be expanded. The rule does not change irrespective of the search problem being solved. These strategies do not take into account any domain specific information about the particular search problem. Popular uninformed search techniques:

• Breadth-First,
• Uniform-Cost,
• Depth-First,
• Depth-Limited, and
• Iterative-Deepening search
Selecting vs. Sorting

Any selection rule can be achieved by employing an appropriate ordering of the frontier set.

A simple equivalence we will exploit
- Order the elements on the frontier.
- Always select the first element.

Breadth First

Place the successors of the current state at the end of the frontier.

Example:
- let the states be the positive integers \( \{0, 1, 2, \ldots \} \)
- let each state \( n \) have as successors \( n+1 \) and \( n+2 \)
  - E.g. \( S(1) = \{2, 3\}; S(10) = \{11, 12\} \)
  - Start state 0
  - Goal state 5

Breadth First Example.

\[(0<>)
(1,2)
(2,2,3)
(2,3,3,4)
(3,3,4,3,4)
(3,4,3,4,4,5)
\]

Breadth First Properties

Measuring time and space complexity.
- \( b \) = maximum number of successors of any state.
- \( d \) = number of actions in the shortest solution.

Breadth-first search

Breadth-first search on a single binary tree

Breadth First Properties

Completeness?
- The length of the path from the initial state to the expanded state must increase monotonically.
  - we replace each expanded state with states on longer paths.
  - All shorter paths are expanded prior before any longer path.
  - Hence, eventually we must examine all paths of length \( d \), and thus find the shortest solution.
Breadth First Properties

Time Complexity?

1 + b + b² + b³ + ... + b^(d-1) + b^d = O(b^d+1)

Space Complexity?

• O(b^d+1): If goal node is last node at level d, all of the successors of the other nodes will be on the frontier when the goal node is expanded. $b(b^d - 1)$

Optimality?

• Will find shortest length solution
• Least cost solution?

Space complexity is a real problem.

E.g., let $b=10$, and say 1000 nodes can be expanded per second and each node requires 100 bytes of storage:

<table>
<thead>
<tr>
<th>Depth</th>
<th>Nodes</th>
<th>Time</th>
<th>Memory</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1 millsec.</td>
<td>100 bytes</td>
</tr>
<tr>
<td>6</td>
<td>10^6</td>
<td>18 mins.</td>
<td>1.1 GB</td>
</tr>
<tr>
<td>8</td>
<td>10^8</td>
<td>31 hrs.</td>
<td>11 GB</td>
</tr>
</tbody>
</table>

• Run out of space long before we run out of time in most applications.

Uniform-Cost Search

Keep "Frontier" or "OPEN" ordered by increasing cost of the path. Always expand the least cost path.

Identical to breadth first if each action has the same cost.

Uniform-Cost Properties

Completeness?

• If each transition has costs ≥ $\varepsilon > 0$.
• The previous argument used for breadth first search holds; the cost of the path represented by each node n chosen to be expanded must be non-decreasing.

Optimality?

• Finds optimal solution if each transition has cost ≥ $\varepsilon > 0$.
• Explores paths in the search space in increasing order of cost. So must find minimum cost path to a goal before finding any higher costs paths.
Let us prove Optimality more formally. We will reuse this argument later on when we examine Heuristic Search.

**Lemma 1.**
Let \( c(n) \) be the cost of node \( n \) on OPEN (cost of the path represented by \( n \)). If \( n_2 \) is expanded IMMEDIATELY after \( n_1 \) then \( c(n_1) \leq c(n_2) \).

Proof: there are 2 cases:

a. \( n_2 \) was on OPEN when \( n_1 \) was expanded:
We must have \( c(n_1) \leq c(n_2) \) otherwise \( n_2 \) would have been selected for expansion rather than \( n_1 \).

b. \( n_2 \) was added to OPEN when \( n_1 \) was expanded:
Now \( c(n_1) < c(n_2) \) since the path represented by \( n_2 \) extends the path represented by \( n_1 \) and thus cost at least \( c \) more.

**Lemma 2.**
When node \( n \) is expanded every path in the search space with cost strictly less than \( c(n) \) has already been expanded.

Proof:
- Let \( n_0 = \text{Start} \).
- Let \( n_k = \langle \text{Start}, s_1, \ldots, s_k \rangle \) be a path with cost less than \( c(n) \), i.e., \( c(n_k) < c(n) \).
- Let \( n_i \) be the last node on this path that has been expanded:
  - \( \langle \text{Start}, s_1, \ldots, s_i, s_{i+1}, \ldots, s_k \rangle \).
- So, \( n_i \) must still be on the frontier. Also \( c(n_i + 1) < c(n) \) since the cost of the entire path to \( n_k \), \( c(n_k) < c(n) \).
- But then uniform-cost would have expanded \( n_{i+1} \) not \( n \).
- So when node \( n \) is expanded, every node on this path must already be expanded, i.e., the path represented by \( n_k \) has already been expanded. QED

**Lemma 3.**
The first time uniform-cost expands a node \( n \) terminating at state \( S \), it has found the minimal cost path to \( S \) (it might later find other paths to \( S \) but none of them can be cheaper).

Proof:
- All cheaper paths have already been expanded, none of them terminated at \( S \).
- All paths expanded after \( n \) will be at least as expensive, so no cheaper path to \( S \) can be found later.

So, when a path to a goal state is expanded the path must be optimal (lowest cost).

**Uniform-Cost Properties**

- Time and Space Complexity?
  - \( O(|S|^2 \cdot \epsilon^{-1}) \) where
    - \( |S| \) is the cost of the optimal solution
    - \( \epsilon \) is the minimal cost of an action
    - each transition has costs \( \geq \epsilon > 0 \).
  - There may be many paths with cost \( \leq C^* \): there can be as many as \( b^{d/\epsilon} \) paths of length \( d \) in the worst case.
  - Paths with cost lower than \( C^* \) can be as long as \( C^*/\epsilon \) (why no longer?), so might have \( b^{d/\epsilon^2} \) paths to explore before finding an optimal cost path.
Uniform Cost Search Overview

Paths in the State Space ordered by cost

<table>
<thead>
<tr>
<th>Path</th>
<th>Cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>&lt;S&gt;</td>
<td>0</td>
</tr>
<tr>
<td>&lt;S, a&gt;</td>
<td>1.0</td>
</tr>
<tr>
<td>&lt;S, b&gt;</td>
<td>1.0</td>
</tr>
<tr>
<td>&lt;S, a, c&gt;</td>
<td>1.5</td>
</tr>
<tr>
<td>&lt;S, d&gt;</td>
<td>2.0</td>
</tr>
</tbody>
</table>

Note cost is non-decreasing

Uniform Cost Search Overview

Uniform Cost Search expands paths in non-decreasing order of cost. So it only goes down this list.

LEMMA 1
It does not miss any paths on this list. LEMMA 2

Path Cost
---|---
<S> | 0
<S, a> | 1.0
<S, b> | 1.5
<S, a, b, c> | 3.0
<S, a, b, d, e> | 4.0

§ If <S, a, b, d, e> is expanded next, <S, a, b, c> must already been expanded. If not it or <S, a, b, c> or <S, a, b> or <S, a> or <S> must be available for expansion (on OPEN) and would be expanded next as they all have lower cost than <S, a, b, d, e>

Uniform Cost Search Overview

Thus working its way down such a list of paths in this order the first path achieving the goal that Uniform cost search finds will be the cheapest way of achieving the goal.

Critical Properties of Search

Completeness: will the search always find a solution if a solution exists?
Optimality: will the search always find the least cost solution? (when actions have costs)
Time complexity: what is the maximum number of nodes than can be expanded or generated?
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Depth-First Search
Depth First Search

Place the successors of the current state at the front of the frontier.

Therefore always expands the deepest node in the frontier/OPEN

Depth First Properties

Completeness?

• Infinite paths? Cause incompleteness!

• Prune paths with duplicate states?
  We get completeness if state space is finite

Optimality?

No!

Depth-First Properties

Depth-First Backtrack Points = unexplored siblings of nodes along current path.

• These are the nodes that remain on open after we extract a node to expand.

Space Complexity?

• $O(b^d)$, linear space!
  • Only explore a single path at a time.
  • The Frontier/OPEN only contains the deepest node on the current path along with the backtrack points.
  • A significant advantage of DFS

Depth First Search Example

(Applied to the example of Breadth First Search)

$\{0\}$
$\{1,2\}$
$\{2,3,2\}$
$\{3,4,3,2\}$
$\{4,5,4,3,2\}$
$\{5,6,4,3,2\}$
...

Depth First Search

Time Complexity?

• $O(b^m)$ where $m$ is the length of the longest path in the state space.

• Very bad if $m$ is much larger than $d$ (shortest path to a goal state), but if there are many solution paths it can be much faster than breadth first. (Can by good luck bump into a solution quickly).

Depth First Properties

Completeness?

• Infinite paths? Cause incompleteness!

• Prune paths with duplicate states?
  We get completeness if state space is finite

Optimality?

No!

Depth First Search

Place the successors of the current state at the front of the frontier.

Therefore always expands the deepest node in the frontier/OPEN

Figure 3.16 FILES: figures/dfs-progress-noblack.eps (Tue Nov 3 13:30:55 2009).

Depth-first search on a binary tree. The unexplored region is shown in lig...
Depth-Limited Search

Breadth first has computational, especially, space problems. Depth first can run off down a very long (or infinite) path.

Depth limited search
- Perform depth first search but only to a pre-specified depth limit L.
- No node on a path that is more than L steps from the initial state is placed on the Frontier.
- We "truncate" the search by looking only at paths of length L or less.

Now infinite length paths are not a problem.
But will only find a solution if a solution of length ≤ L exists.

Depth Limited Search

DLS(Frontier, Successors, Goal?)
If Frontier is empty return failure
Curr = select state from Frontier
If( Goal?(Curr)) return Curr.
If Depth(Curr) < L
Frontier' = (Frontier – {Curr}) U Successors(state)
Else
Frontier' = Frontier – {Curr}
CutOffOccured = TRUE.
return DLS(Frontier', Successors, Goal?)

Depth Limited Search Example

Iterative Deepening Search

Take the idea of depth limited search one step further.
Solve the problems of depth-first and breadth-first by extending depth limited search
Starting at depth limit L = 0, we iteratively increase the depth limit, performing a depth limited search for each depth limit.
Stop if a solution is found, or if the depth limited search failed without cutting off any nodes because of the depth limit.
- If no nodes were cut off, the search examined all paths in the state space and found no solution ➔ no solution exists.

Iterative Deepening Search
Iterative Deepening Search Example

Iterative Deepening Search Example

Iterative Deepening Search Example

Iterative Deepening Search Example

Iterative Deepening Search Example

Iterative Deepening Search Example

Iterative Deepening Search Properties

Completeness?
- Yes, if a minimal length solution of length d exists. What happens when the depth limit L=d?
- What happens when the depth limit L<d?

Time Complexity?
Iterative Deepening Search Properties

**Time Complexity**
- \((d+1)b^0 + db^1 + ... + (d^b) = O(b^d)\)
- E.g. \(b=4, d=10\)
  - \((11)^{10} = 10^{41} + 9^{41} + ... + 2^{41} + 1^{41} \approx 1,864,131\)
  - \(4^{10} = 1,048,576\) (time to explore bottom level)
  - Most nodes (1 of the 1.8 million) lie on bottom level
  - In fact IDS can be more efficient than breadth first search: nodes at limit are not expanded. BFS must expand all nodes until it expand a goal node.

**Breadth first can explore more nodes than IDS**
- Recall for IDS, the time complexity is \((d+1)b^0 + db^1 + ... + (d^b) = O(b^d)\)
- Recall for BFS, the time complexity is \(1 + b + b^2 + ... + b^d = O(b^d)\)
- E.g. \(b=4, d=10\)
  - For IDS
    - \((11)^{10} = 10^{41} + 9^{41} + ... + 4^{10} \approx 1,864,131\) (nodes generated)
    - \(4^{10} = 1,048,576\) (nodes in bottom level)
    - Most nodes lie on bottom layer.
  - For BFS
    - \(1 + 4 + 4^2 + ... + 4^{10} + 4^{11} \approx 5,592,401\) (states generated)
    - In fact IDS can be more efficient than breadth first search: nodes at limit are not expanded. BFS must expand all nodes until it expand a goal node.

**Space Complexity**
- \(O(bd)\) Still linear!
**Optimal?**
- Will find shortest length solution which is optimal if costs are uniform.
- If costs are not uniform, we can use a "cost" bound instead.
  - Only expand paths of cost less than the cost bound.
  - Keep track of the minimum cost unexpanded path in each depth first iteration, increase the cost bound to this on the next iteration.
  - This can be very expensive. Need as many iterations of the search as there are distinct path costs.

Path/Cycle Checking

**Path Checking**
Recall paths are stored on the frontier (this allows us to output the solution path).
If \(\langle s, n_1, ..., n_k \rangle\) is a path to node \(n_k\), and we expand \(n_k\) to obtain child \(c\), we have
\(\langle s, n_1, ..., n_k, c \rangle\)
As the path to "c".

**Path checking:**
- Ensure that the state \(c\) is not equal to the state reached by any ancestor of \(c\) along this path.
- That is, paths are checked in isolation!
**Example: Arad to Neamt**

![Graph showing the path from Arad to Neamt](image)

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**Path Checking Example**

![Path Checking Example](image)

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**Path Checking Example**

![Path Checking Example](image)

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**Cycle Checking**

Cycle Checking (or “multiple path checking”)
- Keep track of all states previously expanded during the search using a closed list.
- When we expand node $n_i$ to obtain child $c$:
  - Ensure that $c$ is not equal to any previously expanded state.
  - This is called cycle checking, or multiple path checking.
- Why can’t we utilize this technique with depth-first search?
  - If we modify depth-first search to do cycle checking what happens to space complexity?

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**Cycle Checking Example**

![Cycle Checking Example](image)

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**Cycle Checking**

High space complexity, only useful with breadth first search. There is an additional issue when we are looking for an optimal solution:
- With uniform-cost search, we still find an optimal solution:
  - The first time uniform-cost expands a state it has found the minimal cost path to it.
  - This means that the nodes rejected by cycle checking can’t have better paths.
- We will see later that we don’t always have this property when we do heuristic search.
Cycle Checking Example (BFS)