1. Consider the following algorithm:

```
func(n):
    # Pre: n is a natural number
    x = 0
    i = 0
    while i < n:
        i = i + 1
        x = x + i
    return x
```

(a) State preconditions and postconditions for this algorithm.

Postconditions: $x = \sum_{i=0}^{n} i$

(b) Use induction to prove the loop invariants $i \leq n$ and $x = \sum_{j=0}^{i} j$ for the while loop.

Let $k \geq 0$. Assume $H(k): i_k \leq n$ and $x_k = \sum_{j=0}^{i_k} j$.

Want to show that $i_{k+1} \leq n$ and $x_{k+1} = \sum_{j=0}^{i_{k+1}} j$

Case: There is no $k + 1$th iteration. Then $i_{k+1} = i_k$ and $x_{k+1} = x_k$, so the loop invariant holds.

Case: There is a $k + 1$th iteration of the loop.

Then, $i_k$ was such that the loop test passed, i.e. $i_k < n$. Thus, $i_{k+1} = i_k + 1 \leq n$.

$x_{k+1} = x_k + i_{k+1}$ (since $i = i + 1$ is first)

$= \sum_{j=0}^{i_k} j + i_{k+1}$ (by $H(k)$)

$= \sum_{j=0}^{i_{k+1}} j$

So, the loop invariant holds in all cases.

Base case: Let $k = 0$. $i_0 = 0 \leq n$ because $n \in \mathbb{N}$.

$x_0 = 0 = \sum_{j=0}^{0} j = \sum_{j=0}^{i_0} j$.

So the loop invariant holds in all cases.

(c) Prove that the loop terminates.

Let $E_k = n - i_k$.

Need to show that (1) $E_k \in \mathbb{N}$, $\forall k$ and (2) $E_{k+1} < E_k$, if there is a $k + 1$th iteration.

(1): $i_k, n \in \mathbb{N}$, and $i_k \leq n \ \forall k$ by part (b). Thus, $E_k \in \mathbb{N}$, and $E_k \geq 0, \forall k$.

(2): If there is a $k + 1$th iteration, then $E_{k+1} = n - i_{k+1} = n - (i_k + 1) = n - i_k - 1 < n - i_k = E_k$.

Thus, the loop terminates.
2. Prove that the following function is correct (by showing partial correctness and termination), according to its pre- and postconditions.

```python
def f(A):
    # Pre: A is a list of integers
    # Post: Returns true if and only if there is an even number of positive
    # numbers in A
    even = True
    i = 0
    while i < A.length:
        if A[i] > 0:
            even = not even
        i = i + 1
    return even
```

**Partial Correctness:** Consider the loop invariant $i \leq A.length$ and even is True iff there are an even number of positive numbers in $A[0..i-1]$.

**Proof of loop invariant:** Let $k \geq 0$.

Assume $H(k)$: $i_k \leq A.length$ and $even_k$ is True iff there are an even number of positive numbers in $A[0..i_k-1]$.

Show $H(k) \rightarrow C(k)$: $i_{k+1} \leq A.length$ and $even_{k+1}$ is True iff there are an even number of positive numbers in $A[0..i_{k+1}-1]$.

Case: There is no $k+1$th iteration. Then $i_{k+1} = i_k$ and the loop invariant holds.

Case: There is a $k+1$th iteration. Then the loop condition passed, so $i_k < A.length$ and $i_{k+1} = i_k + 1 \leq A.length$.

If $A[i_{k+1}]$ is non-positive, then $even_{k+1} = even_k$, and by $H(k)$ represents the number of positive numbers in $A[0..i_k]$, which is the same as the number of positive numbers in $A[0..i_{k+1}]$. If $A[i_{k+1}]$ is positive, then $even_k$ is negated. There is one more positive number in $A[0..i_{k+1}]$ than there was in $A[0..i_k]$. By $H(k)$, $even_k$ was True if there were an even number of positive numbers in $A[0..i_k]$, and so there are an odd number of positive numbers in $A[0..i_{k+1}]$, and $even_{k+1}$ is False. A symmetric argument can be made if $even_k$ was False.

**Base Case:** Let $k = 0$. $i_0 = 0 \leq A.length$. $even_0$ is True, because there are 0 positive numbers in an empty subarray.

Thus, in all cases, the loop invariant holds.

The loop terminates when $i \geq A.length$. By the proof of the loop invariant, $i \leq A.length$. So, the loop terminates when $i = A.length$. Thus, by the loop invariant, even represents the number of positive numbers in all of $A$.

**Termination:** Let $E_k = A.length - i$. By loop invariant, $i \leq A.length$. So $E_k \geq 0 \rightarrow E_k \in \mathbb{N}$, $\forall k$.

If there is a $k+1$th iteration, then $E_{k+1} = A.length - i_{k+1} = A.length - (i_k + 1) < A.length - i_k = E_k$.

Thus, the loop terminates.