UNIVERSITY OF TORONTO
Department of Computer Science
CSC2361Y, Introduction to the Theory of Computation, Summer 2017

MIDTERM EXAM
Examination Date: 2017.06.29 Time: 19:00 - 22:00

- Do not turn this page until you have received the signal to start.
- This examination is closed book exam.
- Each question tells you how many points is worth.
- It is your responsibility to write clearly.
- Answer each question directly on the examination paper, in the space provided, and use a ‘blank’ page for rough work. If you need more space for one of your solutions, use one of the ‘blank’ pages and indicate clearly the part of your work that should be marked.
- There is an indication at the right-down corner with the number of the page that you are currently at and of the total number of pages

Student Number:

Student Name:

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1. (10 points) Prove by simple induction the following statement for all natural numbers \( n \):

\[
\sum_{t=0}^{n} 2^t = 2^{n+1} - 1
\]
2. (10 points) Prove by complete induction that \( a_n < 2^n \) for all the natural numbers \( n \geq 2 \). Define the sequence of integer \( a_i \) as follows:

\[
a_i = \begin{cases} 
2, & \text{if } 0 \leq i \leq 2 \\
\ a_{i-1} + a_{i-2} + a_{i-3}, & \text{if } i > 2
\end{cases}
\]
3. (15 points) Answer the following three questions about induction:
   a) (5 points) State the well-ordering principle.
   b) (5 points) State the principle of simple induction.
   c) (5 points) Prove that well-ordering implies simple induction.
(continue here question 3)
4. (10 points) Define the set $S$ to be the smallest set such that:

- $(1, 0) \in S$
- if $(x, y) \in S$ then $(x + 1, x + y) \in S$

Prove that for every integer $n \geq 1$, the tuple $\left(n, \frac{n(n-1)}{2}\right)$ is in $S$. 
5. (15 points) Consider the following recursive linear search algorithm.

\begin{verbatim}
REC-LIN-SEARCH(A, i, x):
    if i >= len(A):
        return False
    else:
        return A[i] == x or REC-LIN-SEARCH(A, i + 1, x)
\end{verbatim}

a) (5 points) Define a recurrence $T(n)$ for the worst-case runtime of REC-LIN-SEARCH(A, i, x), where $n = len(A) - i$.

b) (5 points) Use unwinding to find a closed form for $T(n)$. Based on the closed form, state a conjecture for $f(n)$ such that $T(n) \in \Theta(f(n))$.

c) (5 points) Prove the bound on $T(n)$ you gave in part (b) is correct.
(continue here question 5)
6. (20 points) Answer the following three questions about Divide-and-Conquer algorithms:

a) (5 points) State the master theorem.

b) (10 points) Informally describe a divide-and-conquer algorithm to find the index of a local maximum element in list of integers $A$. Your algorithm should have complexity strictly less than $O(n)$. An element of a list is a local maximum if it is greater or equal to its neighbor(s).
   Hint: The list $[1, 2, 3, 2, 4, 7, 5]$ has two local maximums, 3 and 7 with indexes 2 and 5 respectively. Finding either is a correct answer.

c) (5 points) Use the master theorem to find the worst case complexity of your algorithm from (b).
(continue here question 6)
Scratch Paper
Scratch Paper