Strong Induction
recall

- use all resources available to you
  - before it becomes too late!
- what resources?
  - office Hours:
    - M 2-3:30 in PT286C, W 2-4 BA4222, F 3:30-4:30 BA4270
  - the course page and our section page
  - the CS Help Centre
  - the course forum
  - study groups and Peer Instruction
  - email ahchinaei @ cs.toronto.edu
review

❖ Last week
  ▪ Simple Induction
    • AKA: Mathematical Induction or Principle of Mathematical Induction
  ▪ 17 examples

❖ This week
  ▪ Strong Induction
    • AKA: Complete Induction or Second Principle of Mathematical Induction

❖ Next week
  ▪ Structural Induction
  ▪ Well Ordering
Simple Induction

- It's a rule of inference:

\[
\begin{array}{c}
P(b) \\
P(k) \rightarrow P(k+1) \\
P(n) \\
\end{array} \quad \forall k \geq b \in \mathbb{N} \quad \forall n \geq b \in \mathbb{N}
\]

- After all,
  - To show that all domino pieces fall over, we should show that
    1) there is a starting point, i.e., \( P(b) \) holds
    and 2) all pieces are set in a well order such that
      falling of piece \( k \) implies falling of piece \( k+1 \)
      i.e., and \( P(k) \rightarrow P(k+1) \) holds too.
**yet another example**

- **Example 19.** A student who went to office hour 01 has provided the following claim and proof. Is it valid?

- **Conjecture:** Doubts in csc236 can be clarified by further discussion each week (e.g., going to the week’s office hours).
  - Let $P(n)$ denotes $d_n$—read doubts of week $n$—can be clarified by further discussion.
  - Proof by simple induction.
  - **Basis step:** $P(1)$ holds as new doubts were clarified in office hours of week 01.
  - **Inductive step:** We assume that doubts of week $k$ can be clarified by further discussion in that week. We need to show that doubts of week $k+1$ can be clarified too. There are two cases: doubts of week $k+1$ are either from week $k$ (that can be clarified by further discussion, based on the I.H.) or they are new doubts (basis step). This completes the inductive step.
  - Therefore, by simple induction, all doubts in csc236 can be clarified by further discussion.

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proof by strong induction

- **recipe:**
  - to prove that $P(n)$ is true for all natural numbers $n$, we should demonstrate these steps:
    - **Proof method:** “strong induction”
    - **Basis step:** show that $P(n)$ is true for some starting point(s), usually 0 or 1 but not always
    - **Inductive step:** show that $P(k) \rightarrow P(k + 1)$ is true for all natural numbers $k$ greater than the starting point.
      - to complete the inductive step, assume $H$ —i.e., $P(i)$— holds for all $i$’s where $b \leq i \leq k$ for an arbitrary natural number $k$, show that $C$ must be true.
revisit: proof by simple induction

fall of the $k+1$th piece is implied by fall of the previous piece, $k$

In the Inductive Step, we show that:

$$f(d_k) \text{ implies } f(d_{k+1})$$
proof by strong induction

fall of the $k+1^{th}$ piece is implied by fall of all previous pieces, $b..k$

In the Inductive Step, we show that:

$$f(d_b) \land f(d_{b+1}) \land .. \land f(d_k) \text{ implies } f(d_{k+1})$$
**Example 20:**

\( P(n) \): \( n \) can be written as product of prime numbers; \( \forall n \geq 2 \in \mathbb{N} \).

*scratch work*

<table>
<thead>
<tr>
<th>( n )</th>
<th>products</th>
<th>( P(n) )</th>
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... ... ...

Strong Induction 2-9
Example 20: \( P(n) \): \( n \) can be written as product of primes; \( \forall n \geq 2 \in \mathbb{N} \).

Proof by Strong Induction

Basis step: \( P(2) \) holds because 2 is prime.

Inductive step:

Inductive Hypothesis: Assume \( P(i) \) holds for all \( i \in \mathbb{N} \) where \( 2 \leq i \leq k \) for any arbitrary fixed \( k \in \mathbb{N} \), i.e., we assume all numbers less than or equal \( k \) can be written as product of primes.

We need to show that \( P(k+1) \) holds too, i.e., \( k+1 \) can be written as product of primes too.

There are two cases: if \( k+1 \) is prime, we are done as \( P(k+1) \) holds.

If \( k+1 \) is not prime, it’s composite and can be written as:
Example 20: $P(n)$: $n$ can be written as product of primes; $\forall n \geq 2 \in \mathbb{N}$.

If $k+1$ is not prime, it's composite and can be written as:

$$k+1 = m \cdot n \text{ where } m, n \in \mathbb{N} \text{ and } 2 \leq m, n < k+1, \text{ i.e., } 2 \leq m, n \leq k$$

By the Inductive hypothesis, $m$ and $n$ each can be written by product of primes. Therefore, $k+1$ is a product of primes, i.e., $P(k+1)$ holds too.

This completes the inductive step.

Therefore, by strong induction, it proves that $\forall n \geq 2 \in \mathbb{N}$, $n$ can be written as product of primes.
Example 21:

\( P(n) \): any postage \( n \) that is 18 cents or more can exactly be stamped using just 4-cent and 7-cent stamps.

*scratch work*

<table>
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<tr>
<th>( n )</th>
<th>stamps</th>
<th>( P(n) )</th>
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<tbody>
<tr>
<td>18</td>
<td>1<em>4+2</em>7</td>
<td>✓</td>
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\[ \ldots \quad \ldots \quad \ldots \]
Example 21: Proof by Strong Induction

$P(n): n$ can be written as $a*4 + b*7; \ \forall n \geq 18 \in \mathbb{N}.$
Example 21: \[ P(n): n \text{ can be written as } a*4 + b*7; \quad \forall n \geq 18 \in \mathbb{N}. \]
Example 22:

Make a conjecture to specify the minimum number of breaks to break a chocolate bar to all chocolate squares. Proof your claim.

scratch work
Example 22:
Example 22:
Example 23:

Let \( f(n) = \begin{cases} 2^n & n = 1 \\ f^2(\lfloor \sqrt{n} \rfloor) + 2f(\sqrt{n}) & n > 1 \end{cases} \), prove \( f(n) \) is a multiple of 8.

scratch work
Example 23:
Example 24:

Prove that every simple polygon with \( n \) sides can be composed of \( n-2 \) triangles.

*scratch work*
Example 24: $P(n)$: An $n$-sided polygon can be triangulated to $n-2$ triangles; $\forall n \geq 18 \in \mathbb{N}$. 
Example 24: P(n): An \textit{n-sided} polygon can be triangulated to \textit{n}-2 triangles; \( \forall n \geq 18 \in \mathbb{N} \).
strong induction recipe (revisited)

0. write out the claim as: “Let \( P(n) \) denote the claim in terms of \( n \)” follow next steps to show that \( P(n) \) holds \( \forall n \geq b \in \mathbb{N} \), where \( b \) is staring point(s)

1. write out “Proof method: strong induction”

2. write out “Basis step:” followed by reasoning that \( P(b) \) is true

3. write out “Inductive step:”
   3.1. write out “Inductive hypothesis: we assume \( P(i) \) is true \( \forall i, b \leq i \leq k \)” where \( P(i) \) is the claim in terms of \( i \)
   3.2. reason that \( P(k+1) \) is true
       note 1: in your reasoning here, you must use the inductive hypothesis
       note 2: be sure your reasoning is true for any \( k \geq b \), including \( k=b \)
   3.3. write out “This completes the inductive step”

4. write out “This proves \( P(n) \) is true for \( \forall n \geq b \in \mathbb{N} \)” where \( P(n) \) is the claim in terms of \( n \)

5. Indicate end of proof by “\( \Box \)”. 

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