The aim of this assignment is to give you some practice with various forms of induction. For each question below you will present a proof by induction. For full marks you will need to make it clear to the reader that the base case(s) is/are verified, that the inductive step follows for each element of the domain (typically the natural numbers), where the inductive hypothesis is used and that it is used in a valid case.

Your assignment must be typed to produce a PDF document (hand-written submissions are not acceptable). You may work on the assignment in groups of 1, 2, or 3, and submit a single assignment for the entire group on MarkUs.

1. Consider the Fibonacci-esque function $g$:

\[
g(n) = \begin{cases} 
1, & \text{if } n = 0 \\
3, & \text{if } n = 1 \\
g(n-2) + g(n-1) & \text{if } n > 1
\end{cases}
\]

Use complete induction to prove that if $n$ is a natural number greater than 1, then $2^{n/2} \leq g(n) \leq 2^n$.

You may not derive or use a closed-form for $g(n)$ in your proof.

Sample solution: Proof, using complete induction.

**Inductive step:** Let $n$ be a typical natural number greater than 1 and assume $H(n)$: Every natural number $i \in \{2, \ldots, n-1\}$ satisfies $2^{i/2} \leq g(i) \leq 2^i$.

**Show that inductive conclusion follows:** We'll derive $C(n)$: $2^{n/2} \leq g(n) \leq 2^n$.

**Base cases:** $1 < n < 4$: $g(2) = 4$ and $g(3) = 7$ # by the definition of $g(2)$ and $g(3)$.

\[
2^{2/2} = 2 \leq 4 = g(2) \leq 4 = 2^2 \quad \text{and} \quad 2^{3/2} = 2\sqrt{2} \leq 7 = g(3) \leq 8 = 2^3
\]

$C(2)$ and $C(3)$ follow from our assumptions in this case.

**Case $n \geq 4$:** By assumptions $H(n-2)$ and $H(n-1)$ # $n \geq 4$ implies $2 \leq n - 2, n - 1 < n$:

\[
2^{(n-2)/2} \leq g(n-2) \leq 2^{n-2} \quad \text{and} \quad 2^{(n-1)/2} \leq g(n-1) \leq 2^{n-1}.
\]

Substituting these inequalities into the definition of $g(n)$ # by definition of $g(n), n \geq 4 > 0$:

\[
g(n) = g(n-2) + g(n-1) \geq 2^{(n-2)/2} + 2^{(n-1)/2} = (1 + \sqrt{2})2^{(n-2)/2} \geq 2 \times 2^{(n-2)/2} = 2^{n/2}
\]

\[
g(n) = g(n-2) + g(n-1) \leq 2^{n-2} + 2^{n-1} = (1 + 2)2^{n-2} \leq 2^2 \times 2^{n-2} = 2^n
\]

$C(n)$ follows from our assumptions in this case.

In all cases $H(n)$ implies $C(n)$. 
2. Suppose \( B \) is a set of binary strings where each binary string is of length \( n \). \( n \) is positive (greater than 0), and no two strings in \( B \) differ in fewer than 2 positions. Use simple induction to prove that \( B \) has no more than \( 2^{n-1} \) elements.

Sample solution: Proof, using simple induction.

verify base: There are two binary strings of length 1: "0" and "1", and they differ from each other in exactly one position (i.e. fewer than 2). That means that the only sets of binary strings of length 1 that contain no pairs that differ in fewer than 2 positions are \{"1"\}, \{"0"\}, and \{\}, which each have no more than 1 = 2\(^{1-1}\) elements, verifying the claim in this case.

inductive step: Let \( n \) be a typical natural number greater than 0. Assume \( H(n) \): any set of binary strings of length \( n \) containing no pairs that differ in fewer than 2 positions must have no more than \( 2^{n-1} \) elements.

derive conclusion \( C(n) \): We must show that from \( H(n) \) follows \( C(n) \): any set of binary strings of length \( n + 1 \) containing no pairs that differ in fewer than 2 positions must have no more than \( 2^n \) strings.

Let \( B \) be an arbitrary set of binary strings of length \( n + 1 \) that contains no pairs that differ in fewer than 2 positions.

Let \( B_1 \) be the subset of \( B \) consisting of those elements with 1 in the first position, and \( B_2 \) be the subset of \( B \) consisting of those elements of \( B \) with 0 in the first position. \( B_1 \) and \( B_2 \) partition \( B \), since every element of \( B \) has either a 1 or a 0 in the first position, and no element of \( B \) has both a 1 and a 0 in the first position.

From \( B_1 \) construct \( B'_1 \), consisting of the strings of \( B_1 \) with the leading 1 removed. Similarly, from \( B_2 \) construct \( B'_2 \), consisting of the strings of \( B_2 \) with the leading 0 removed.

Sets \( B'_1 \) and \( B'_2 \) contain strings of length \( n \), and contain no pairs that differ in fewer than 2 positions, since removing the leading 1s or 0s cannot change the number of positions in which elements differ. By assumption \( H(n) \), both \( B'_1 \) and \( B'_2 \) have no more than \( 2^{n-1} \) elements each.

Elements of \( B_1 \) are in 1-1 correspondence with those of \( B'_1 \), since you can transform one into the other by prepending a leading 1, or removing a leading 1. Similarly elements of \( B_2 \) are in 1-1 correspondence with those of \( B'_2 \). \(|B_1| = |B'_1| \) and \(|B_2| = |B'_2|\), since they are in 1-1 correspondence.

\(|B| = |B_1| + |B_2| \leq 2^{n-1} + 2^{n-1} = 2^n\), since \( B_1 \) and \( B_2 \) partition \( B \). This is what \( C(n) \) claims.

3. Define \( T \) as the smallest set of strings such that:

(a) "b" \( \in T \)

(b) If \( t_1, t_2 \in T \), then \( t_1 + "ene" + t_2 \in T \), where the + operator is string concatenation.

Use structural induction to prove that if \( t \in T \) has \( n "b" \) characters, then \( t \) has \( 2n - 2 "e" \) characters.

Sample solution: Proof, using structural induction.

verify basis: "b" \( \in T \# \) from definition. "b" has 1 character "b" and \( 2(1) - 2 = 0 "e" \) characters. This verifies the claim for the basis.

inductive step: Let \( t_1, t_2 \in T \) and assume \( H(\{t_1, t_2\}) \): If \( t_1 \) has \( n_1 "b" \) characters and \( t_2 \) has \( n_2 "b" \) characters, then \( t_1 \) has \( 2n_1 - 2 "e" \) characters and \( t_2 \) has \( 2n_2 - 2 "e" \) characters.

show that inductive conclusion follows from assumptions: We'll derive \( C(t_1 + "ene" + t_2) \): If \( t_1 + "ene" + t_2 \) has \( n_{1,2} "b" \) characters, then it has \( 2n_{1,2} - 2 "e" \) characters.

\( t_1 + "ene" + t_2 \in T \# \) by definition of \( T \), where + is string concatenation.
Let $n_1, m_1$ be the number of "b" (respectively "e") characters in $t_1$, and $n_2, m_2$ be the number of "b" (respectively "e") characters in $t_2$. $t_1 + "ene" + t_2$ has $n_1 + n_2$ "b" characters # Concatenating "ene" doesn't increase the number of "b" characters.

Let $n_{1,2}$ be the number of "b" characters in $t_1 + "ene" + t_2$. Then $n_{1,2} = n_1 + n_2$ # since no "b" characters are added by concatenating "ene".

$t_1 + "ene" + t_2$ has $2n_1 - 2 + 2n_2 - 2 + 2$ "e" characters # by assumptions $H(t_1), H(t_2)$, and two "e" characters in "ene".

Summing up $t_1 + "ene" + t_2$ has $2n_1 - 2 + 2n_2 - 2 + 2(n_1 + n_2) - 2 + 2 = 2n_1 + 2n_2 - 2$ "e" characters. Conclusion $C(t_1 + "ene" + t_2)$ follows in this case.

4. On page 79 of the Course Notes the quantity $\phi = (1 + \sqrt{5})/2$ is shown to be closely related to the Fibonacci function. You may assume that $1.61803 < \phi < 1.61804$. Complete the steps below to show that $\phi$ is irrational.

(a) Show that $\phi(\phi - 1) = 1$.

Sample solution: Substituting the expression for $\phi$:

$$\phi(\phi - 1) = \left(\frac{1 + \sqrt{5}}{2}\right) \left(\frac{\sqrt{5} - 1}{2}\right) = \frac{4}{4} = 1$$

(b) Rewrite the equation in the previous step so that you have $\phi$ on the left-hand side, and on the right-hand side a fraction whose numerator and denominator are expressions that may only have integers, + or -, and $\phi$. There are two different fractions, corresponding to the two different factors in the original equation's left-hand side. Keep both fractions around for future consideration.

Sample solution: I can choose to divide 1 by either $\phi$ or $\phi - 1$, yielding:

$$\phi = \frac{1}{\phi - 1} \quad \phi = \frac{1 + \phi}{\phi}$$

(c) Assume, for a moment, that $\phi$ is the ratio of two natural numbers. Let $m, n \in \mathbb{N}$ such that $\phi = n/m$. Re-write the right-hand side of both equations in the previous step so that you end up with fractions whose numerators and denominators are expressions that may only have integers, + or -, $m$ and $n$.

Sample solution: Substitute $n/m$ for $\phi$ on the right-hand side, and then simplify:

$$\phi = \frac{1}{\phi - 1} \quad \Rightarrow \quad \phi = \frac{m}{n - m} \quad \phi = \frac{1 + \phi}{\phi} \quad \Rightarrow \quad \frac{m + n}{n}$$

(d) Use your assumption from the previous part to construct a non-empty subset of the natural numbers that contains $m$. Use the Principle of Well-Ordering, plus one of the two expressions for $\phi$ from the previous step to derive a contradiction.

Sample solution: Let $F \subseteq \mathbb{N}$ be defined by:

$$F = \{m' \in \mathbb{N} \mid \exists n' \in \mathbb{N}, \phi = n'/m'\}.$$ 

By assumption in (c), $F$ is non-empty, since it has at least one member, $m$. By PWO $F$ has a smallest element, let it be $m_0$, with its corresponding $n_0$ so that $m_0, n_0 \in \mathbb{N}$ and $\phi = n_0/m_0$.
Rewriting the equation for $\phi$ and using the assumption $1.61803 < \phi < 1.61804$ yields:

$$\phi = \frac{n_0}{m_0}$$

$$\phi m_0 = n_0 \quad \# \text{multiply both sides by } m_0$$

$$m_0 < n_0 < 2m_0 \quad \# \text{multiply } 1.61803 < \phi < 1.61804 \text{ by } m$$

$$0 < n_0 - m_0 < m_0 \quad \# \text{subtract } m_0 \text{ from both inequalities.}$$

$n_0 - m_0 \in \mathbb{N} \quad \# \text{integers closed under } -$ and difference is non-negative.

$n_0 - m_0 \in F$, since $\phi = n_0/m_0 = m_0/(n_0 - m_0)$ and $n_0 - m_0 < m_0$.

Contradiction $\rightarrow \leftarrow$. $m_0$ is the smallest element of $F$, by construction.

(e) Combine your assumption and contradiction from the previous step into a proof that $\phi$ cannot be the ratio of two natural numbers. Extend this to a proof that $\phi$ is irrational.

Sample solution: Proof (by contradiction) that $\phi$ is irrational.

Assume, for the sake of contradiction, that $\phi$ is rational.

Let $z_1, z_2 \in \mathbb{Z}, \phi = z_1/z_2 \quad \# \text{by definition of } \phi \text{ is rational}$

Let $n, m \in \mathbb{N}, m/n = \phi. \quad \# \text{Since } \phi = z_1/z_2 > 0 \text{ the numerator and denominator have the same sign. If } z_1, z_2 > 0, \text{ let } n = z_1, m = z_2. \text{ Otherwise, if } z_1, z_2 < 0, \text{ let } n = -z_1, m = -z_2.$

Contradiction $\rightarrow \leftarrow$. From the previous part, there are no natural numbers $m, n$ such that $\phi = m/n$.

$\phi$ is irrational, since assuming otherwise leads to a contradiction.

5. Consider the function $f$, where $3 \div 2 = 1$ (integer division, like $3\div2$ in Python):

$$f(n) = \begin{cases} 1 & \text{if } n = 0 \\ f^2(n \div 3) + 3f(n \div 3) & \text{if } n > 0 \end{cases}$$

Use complete induction to prove that for every natural number $n$ greater than 2, $f(n)$ is a multiple of 7. NB: Think carefully about which natural numbers you are justified in using the inductive hypothesis for.

Sample solution: Proof by complete induction.

Inductive step: Let $n$ be a typical natural number greater than 2, and assume $H(n):$ that $f(i)$ is a multiple of 7 for natural numbers $2 < i < n$.

Show that inductive conclusion follows: We’ll derive $C(n): f(n)$ is a multiple of 7.

Base case $2 < n < 6$: $n > 0$, so by the definition of $f(n)$:

$$f(n) = f^2(n \div 3) + 3f(n \div 3) = f^2(1) + 3f(1) = 28 \quad \# f(1) = 4 \text{ from definition}$$

28 is a multiple of 7, so $C(n)$ follows in this case.

Base case $6 \leq n < 9$: $n > 0$, so by the definition of $f(n)$:

$$f(n) = f^2(n \div 3) + 3f(n \div 3) = f^2(2) + 3f(2) = 28 \quad \# f(2) = 4 \text{ from definition}$$

28 is a multiple of 7, so $C(n)$ follows in this case.

Case $n \geq 9$: $n > n \div 3 > 2$, so by assumption $H(n \div 3)$ we know that $f(n \div 3)$ is a multiple of 7.

Let $k \in \mathbb{N}$ be a natural number such that $f(n \div 3) = 7k$.

$$f(n) = f^2(n \div 3) + 3f(n \div 3) = 49k^2 + 21k = 7(7k^2 + 3k)$$

$7(7k^2 + 3k)$ is a multiple of 7, so the conclusion $C(n)$ is verified in this case.