Todays agenda and announcements


• Reading for next two weeks: Ch.23 (plus some important ideas not discussed in the text)
• This week: 23.1-23.6; next week: 23.7-23.10

Announcements

• Final quiz (quiz 8) scheduled for April 1. Aids allowed for the final exam is the same as for all quizzes and tests; one 8.5 by 11 sheet (2 sides) of handwritten notes are the only aids allowed.
• As in all assignments, quizzes and tests, you will receive 20% credit for any question (or question part) where you explicitly state “I do not know how to answer this question”.
• Last assignment is due March 30 and has been posted.
• Office hours by appointment for next couple weeks.
Voting and Preference Aggregation

- Last time
  - Introduced **social choice**: preference aggregation to make a single “consensus” decision for a group
  - The concept of a **voting rule**:
    - Given: a set $N$ of $n$ voters and a set $A$ of $m$ alternatives
    - Input: a preference profile (a ranking of alternatives by each voter)
    - Output: winning alternative from $A$
    - Also discussed the idea of deriving a *consensus ranking* over $A$
  - Different voting rules (Plurality, Borda, approval, STV, etc.) and properties
    - Different rules give different results on same profiles!
Plurality Voting

- **Plurality voting:**
  - **Input:** rankings of each voter
  - **Winner:** alternative ranked 1\textsuperscript{st} by greatest number of voters
    - number of 1\textsuperscript{st}-place rankings is a’s *plurality score*
    - *complete* rankings not needed, just votes for most preferred alternatives
    - we’ll ignore ties for simplicity
  - This is a most familiar scheme, used widely:
    - locally, provincially, nationally for electing political representatives
  - With only 2 alternatives, often called *majority voting*

- Example preference profile (three alternatives):
  - A > B > C: 5 voters
  - C > B > A: 4 voters
  - B > C > A: 2 voters

- **Winner:** A wins (plurality scores are A: 5; C: 4; B: 2)
The Borda Rule

- **Borda voting rule:**
  - Input: rankings of each voter
  - Borda score for each alternative $a$: $a$ gets $m-1$ points for every 1$^{\text{st}}$-place rank, $m-2$ points for every 2$^{\text{nd}}$-place, etc.
  - Winner: alternative with highest Borda score
  - Used in sports (Heisman, MLB awards), variety of other places
  - Proposed by Jean-Charles, chevalier de Borda in 1770 to elect members to the French Academy of Sciences (also Ramon Llull, 13$^{\text{th}}$ century)

- Example profile (three alternatives, positional scores of 2, 1, 0):
  - $A > B > C$: 5 voters
  - $C > B > A$: 4 voters
  - $B > C > A$: 2 voters

- Winner: $B$ wins (Borda scores are: $B$: 13; $A$: 10; $C$: 10)
  - Notice: more sensitive to the entire range of preferences than plurality is (which ranked $B$ last)
Approval Voting

**Approval Voting**

- **Input:** voters specify a *subset* of alternatives they “approve of”
- Approval score: a point given to a for each approval
  - variant: *k*-approval, voter lists exactly *k* candidates
- **Winner:** alternative with highest approval score
- Used in many informal settings (at UN, Doge of Venice, …)
- Steven Brams a major advocate (see Wikipedia article)

**Example profile (three alternatives, approvals in bold):**

- **A > B > C:** 5 voters (approve of only top alternative)
- **C > B > A:** 4 voters (approve of only top alternative)
- **B > C > A:** 2 voters (approve of top two alternatives)

**Winner:** C wins (approval scores are: C: 6; A: 5; B: 2)
  - Notice: can’t predict vote based on ranking alone!
Positional Scoring (Voting) Rules

- Observe that plurality, Borda, $k$-approval, $k$-veto are all each *positional scoring rules*
- Each assigns a *score* $\alpha(j)$ to each rank position $j$
  - almost always non-increasing in $j$
- The winner is the candidate $a$ with max total score: $\sum_i \alpha(r_i(a))$

<table>
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<tr>
<th></th>
<th>$a(1)$</th>
<th>$a(2)$</th>
<th>$a(3)$</th>
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<tr>
<td><strong>Borda</strong></td>
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<td>2</td>
<td>1</td>
<td>0</td>
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<tr>
<td><strong>2-Approval</strong></td>
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<td><strong>Veto</strong></td>
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<td>1</td>
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<td><strong>and another</strong></td>
<td>10</td>
<td>2</td>
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Which of these is Better?

- Notice that on the same vote profile, plurality, Borda, and approval gave different winners!

- Which is best?
  - hard to say: depends on social objective one is trying to meet
  - common approach: identify *axioms/desirable properties* and try to show certain voting rules satisfy them
    - we will see it is not possible in general!

- Note: all these voting rules have to have some tie breaking breaking rule. In some cases, that rule is simply a flip of the coin. See the tie vote in a recent election in Mississippi. Even with a large number of voters ties can happen.

- Let’s now look at a few more voting rules to get a better sense of things.
There are Hundreds of Voting Rules

- **Single-transferable vote (STV) or Hare system**
  - Round 1: vote for favorite candidate; eliminate candidate with lowest plurality score;
  - Round $t$: if your favorite eliminated at round $t-1$, recast vote for favorite remaining candidate; eliminate candidate with lowest plurality score
  - Round $m-1$: winner is last remaining candidate
    - terminate at any round if plurality score of top candidate is at least $n/2$ (i.e., there is a majority winner)
  - Used: Australia, New Zealand, Ireland, recent NDP convention
    - Needn’t be online: voters can submit rankings once
  - When would this be a bad voting rule?

- **Nanson’s rule**
  - Just like STV, but use Borda score to eliminate candidates
There are Hundreds of Voting Rules

- **Egalitarian (maxmin fairness)**
  - Winner maximizes minimum voter’s rank: \( \text{argmax}_a \min_j (m-r_j (a)) \)

- **Copeland**
  - Let \( W(a,b,r) = 1 \) if more voters rank \( a \succ b \); 0 if more \( b \succ a \); \( \frac{1}{2} \) if tied
  - Score \( s_c(a,r) = \sum_b W(a,b,r) \); winner is \( a \) with max score
    - i.e., *winner is candidate that wins most pairwise elections*

- **Tournament/Cup**
  - Arrange a (usually balanced) tournament tree of pairwise contests
  - Winner is last surviving candidate
  - We’ll discuss this in more detail later
Condorcet Principle

- How would you determine “societal preference” between a pair of alternatives $a$ and $b$?
- A natural approach: run a “pairwise” majority vote: if a *majority* of voters prefer $a$ to $b$, then we say *the group prefers $a$ to $b*
- **Condorcet winner**: an alternative that beats every other in a pairwise majority vote
  - proposed by Marie Jean Antoine Nicolas de Caritat, marquis de Condorcet in 1785
  - if there is a Condorcet winner, it must be unique
  - a rule is **Condorcet-consistent** if it selects the Condorcet winner (if one exists)

- Condorcet winners need not exist (next slide)
  - Moreover, many natural voting rules are not Condorcet consistent (e.g., plurality, Borda, STV are not), but some are: Nanson, Copeland, Cup, etc.
Condorcet Paradox

- **Condorcet paradox:**
  - suppose we use the pairwise majority criterion to produce a societal preference ranking
  - pairwise majority preferences may induce *cycles* in societal ranking (i.e., the preference relation is not transitive)

- **Simple example:**
  - A > B > C: \( m/3 \) voters
  - C > A > B: \( m/3 \) voters
  - B > C > A: \( m/3 \) voters
  - Societal ranking has A > B, B > C, and C > A (!)
  - No clear way to produce a consensus ranking
  - Also evident that this preference profile has no Condorcet winner
Violations of Condorcet Principle

- **Plurality violates Condorcet**
  - 499 votes: $A > B > C$
  - 3 votes: $B > C > A$
  - 498 votes: $C > B > A$
  - plurality chooses $A$; but $B$ is a CW ($B > A$ 501:499; $B > C$ 502:498)

- **Borda violates Condorcet**
  - 3 votes: $A > B > C$
  - 2 votes: $B > C > A$
  - 1 vote: $B > A > C$
  - 1 vote: $C > A > B$
  - Borda chooses $B$ (9 pts) ; but $A$ is a CW ($A > B$ 4:3; $A > C$ 4:3)
  - notice *any* scoring rule (not just Borda) will choose $B$ if scores strictly decrease with rank
Considerable work studies various “axioms” or principles that we might like voting rules to satisfy and asks whether we can devise rules that meet these criteria.

For example, the Condorcet principle is an axiom/property we might consider desirable. We’ve seen some standard voting rules satisfy it, and others do not.

Let’s consider a few more rather intuitive axioms…
Weak Monotonicity

- **Weak monotonicity:** Let $V$ be a set of vote profiles and let $V'$ be identical to $V$ except that one alternative $a$ is ranked higher in some of the votes. Then if $a$ is the winner under voting rule $r$ with profile $V$, it should also be the winner with profile $V'$.

  • That is, if $a$ is the winner under some voting rule given some voter preferences, then $a$ should remain the winner if a few voters raise their ranking of $a$, but everything else is unchanged.

- **STV violates weak monotonicity**
  - 22 votes: $A > B > C$
  - 21 votes: $B > C > A$
  - 20 votes: $C > A > B$
  - A wins (C, then B eliminated)...
  - ... but if anywhere from 2 to 9 voters in the BCA group “promote” A to top of their rankings, C wins (B, then A eliminated)

- **Lot of rules satisfy weak monotonicity** (e.g. plurality, Borda, …)
Independence of Irrelevant Alternatives (IIA)

- *Independence of Irrelevant Alternatives (IIA):* Suppose $V'$ is a vote profile that is different than $V$, but every vote in $V'$ gives the same relative ordering to $a$, $b$, as it does in $V$. Then if $a$ is the winner under a voting rule $r$ given profile $V$, the $b$ cannot be the winner under profile $V'$.
  - In other words, if the votes are changed, but the relative (pairwise) preference for $a$ and $b$ are identical for every voter, then we can’t change the winner from $a$ to $b$.

- **Borda violates IIA (as do quite a few other voting systems):**
  - 3 votes: $A > B > C > D > E$
  - 1 vote: $C > D > E > B > A$ (switch: $C > B > E > D > A$)
  - 1 vote: $E > C > D > B > A$ (switch: $E > C > B > D > A$)
  - C wins under red votes (Borda scores: $C:13$, $A:12$, $B:11$, $D:8$, $E: 6$)
  - … but with the blue switches, $B$ wins (scores $B:14$, $C:13$, …).
  - Winner from $C$ to $B$, despite all paired $B,C$ prefs identical in both cases.
Independence of Irrelevant Alternatives (IIA)

- Another view of IIA: suppose $a$ wins over $b$ in an election. Then we add a new alternative. *Without changing anyone’s relative preferences for $a$ and $b$, suddenly $b$ can win.*

- Consider the following preferences:
  - 100 votes: $\text{Bush} \succ \text{Gore} \succ \text{Nader}$
  - 12 votes: $\text{Nader} \succ \text{Gore} \succ \text{Bush}$
  - 95 votes: $\text{Gore} \succ \text{Nader} \succ \text{Bush}$

  - Run a plurality election with only two candidates, Bush and Gore
    - Gore wins over Bush (plurality score of 107 to 100)
  - At the least minute, Nader enters the race:
    - Bush wins the election now (plurality score of 100 to 95 to 12)
Other Principles

- **Unanimity**: if all \( v \in V \) rank \( a \) first, then \( a \) wins
  - relatively uncontroversial
- **Weak Pareto**: if for all \( v \in V \) rank \( a > b \), then \( b \) cannot win
  - relatively uncontroversial
- **Non-dictatorial**: there is no voter \( k \) s.t. \( a \) is the winner whenever \( k \) ranks \( a \) first (no matter what other voters say)
- **Anonymity**: permuting votes within a profile doesn’t change outcome
  - e.g., if all votes are identical, but provided by “different” voters, result does not change (can’t depend on voter’s identities)
  - implies non-dictatorship
- **Neutrality**: permuting alternatives in a profile doesn’t change outcome
  - i.e., result depends on relative position of an alternative in the votes themselves, not on the identity of the alternative
  - implies non-imposition (i.e. every possible ranking is achievable)
Arrow’s Theorem

- So can we satisfy all (or even some of these axioms)?
- Arrow’s Theorem (1951): Assume at least three alternatives. No voting rule can satisfy IIA, weak Pareto, and non-dictatorship.
  - Most celebrated theorem in social choice
  - Broadly (perhaps too broadly) interpreted as stating there is no good way to aggregate preferences
  - Key point: Arrow’s Theorem is phrased in terms of a rule producing a ranking.

- There are a wide variety of alternative proofs
  - Easley and Kleinberg provide one proof (see Ch. 23.11)
  - An especially simple proof is given the next two slides for anyone interested.